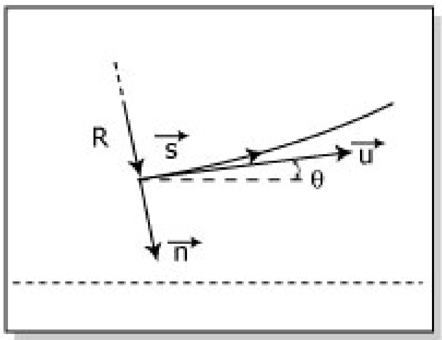
16.512, Rocket Propulsion
Prof. Manuel Martinez-Sanchez
Lecture 4-5: Nozzle Design: Method of Characteristics

<u>The Method of Characteristics (Ideal Gas)</u>

(Ref. Phillip Thompson <u>Compressive Fluid Dynamics</u>, McGraw Hill, 1972, Ch. 9)

2-D or axisymmetric

Homentropic as well as isentropic $\longrightarrow \nabla \times \vec{u} = o$



plus $\nabla \cdot \left(\rho\ \vec{u}\right) = 0$

and $\vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho}\nabla p = o \Rightarrow \nabla\left(\frac{u^2}{2}\right) + \cancelto{0}{\vec{\omega}} \times \vec{u} + \frac{1}{\rho}\nabla p = 0$

Use intrinsic co-ordinates

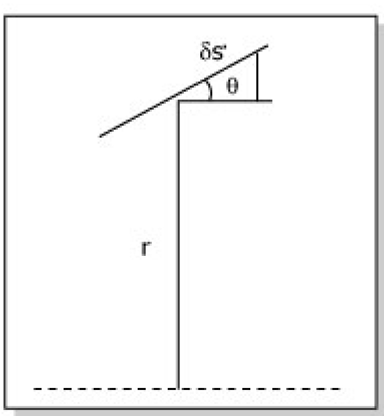
Eq. of motion along $\vec{s}$:
$$\boxed{u\frac{\partial u}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} = o} \qquad (1)$$

Eq. of motion along $\vec{n}$:
$$\frac{u^2}{R} = \frac{1}{\rho}\frac{\partial p}{\partial n} = -\frac{\partial u^2/2}{\partial n}$$

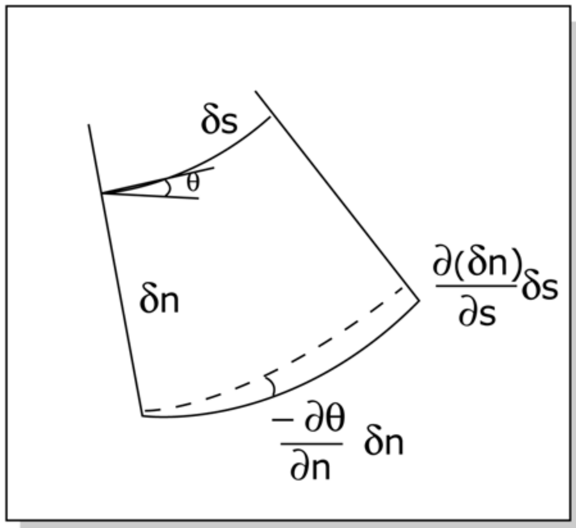
Now $\frac{1}{R} \equiv \frac{\partial \vartheta}{\partial s}$ $\qquad u^2\frac{\partial \vartheta}{\partial s} = -u\frac{\partial u}{\partial n}$ $\qquad \boxed{\frac{\partial u}{\partial n} + u\frac{\partial \vartheta}{\partial s} = o} \qquad (2)$

(good no p in it)

(can also get this from $\nabla \times \vec{u} = o$)



Continuity: $\frac{\partial}{\partial s}\left(\rho\, u\, 2\pi r\ \delta n\right) = o$



$$\frac{1}{\rho}\frac{\partial \rho}{\partial s}+\frac{1}{u}\frac{\partial u}{\partial s}+\frac{1}{r}\frac{\partial r}{\partial s}+\frac{1}{\partial n}\frac{\partial \delta n}{\partial s}=0$$

$$(\delta s)\sin\vartheta=\frac{\partial r}{\partial s}(\partial s) \quad \frac{1}{r}\sin\vartheta \quad \frac{\partial \vartheta}{\partial n}$$

$$\frac{\partial \delta n}{\partial s}ds=\left(-\frac{\partial \vartheta}{\partial n}\delta n\right)\delta s$$

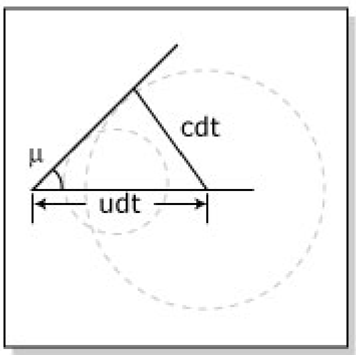
$$\boxed{\frac{1}{\rho}\frac{\partial \rho}{\partial s}+\frac{1}{u}\frac{\partial u}{\partial s}-\frac{\partial \vartheta}{\partial n}=-\frac{\sin\vartheta}{r}}$$

Homentropic: $dp=c^2\,d\rho$ $\quad\left(c^2=\left(\frac{\partial p}{\partial \rho}\right)_s\right)$

so $\qquad \frac{1}{\rho c^2}\frac{\partial p}{\partial s}+\frac{1}{u}\frac{\partial u}{\partial s}-\frac{\partial \vartheta}{\partial n}=-\frac{\sin\vartheta}{r}$

and, from s eq. of motion, $\frac{1}{\rho}\frac{\partial p}{\partial s}=-u\frac{\partial u}{\partial s}$ so you can eliminate $\frac{\partial p}{\partial s}$:

$$-\frac{u}{c^2}\frac{\partial u}{\partial s}+\frac{1}{u}\frac{\partial u}{\partial s}-\frac{\partial \vartheta}{\partial n}=-\frac{\sin\vartheta}{r} \qquad \frac{1}{u}\underbrace{\left(-1+\frac{u^2}{c^2}\right)}_{M^2-1}\frac{\partial u}{\partial s}+\frac{\partial \vartheta}{\partial n}=\frac{\sin\vartheta}{r}$$



$$\frac{M^2-1}{u}\frac{\partial u}{\partial s}+\frac{\partial \vartheta}{\partial n}=\frac{\sin\vartheta}{r} \qquad (4)$$

Introduce the Mach angle $\mu=\sin^{-1}\frac{1}{M}=\tan^{-1}\frac{1}{\sqrt{M^2-1}}$ $\qquad \tan\mu=\frac{1}{\sqrt{M^2-1}}$

Then (2) $\longrightarrow$ $\tan\mu\frac{\sqrt{M^2-1}}{u}\frac{\partial u}{\partial n}+\frac{\partial \vartheta}{\partial s}=0$

And (4) $\longrightarrow$ $\frac{\sqrt{M^2-1}}{u}\frac{\partial u}{\partial s}+\tan\mu\frac{\partial \vartheta}{\partial n}=\frac{\tan\mu\,\sin\vartheta}{r}$

Introduce the Prandtl-Meyer function $\omega(M)$ by

$$d\omega=\sqrt{M^2-1}\,\frac{du}{u} \qquad \text{(to be integrated later)}$$

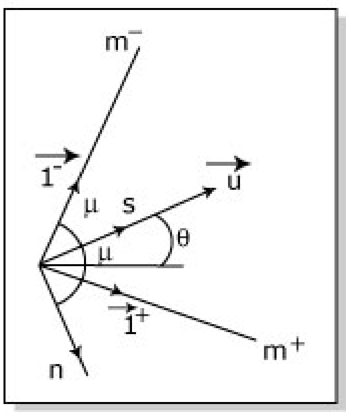
$$\left.\begin{aligned} \tan\mu\frac{\partial\omega}{\partial n}+\frac{\partial\vartheta}{\partial s}&=o \\ \frac{\partial\omega}{\partial s}+\tan\mu\frac{\partial\vartheta}{\partial n}&=\frac{\tan\mu\,\sin\vartheta}{r}\end{aligned}\right\}$$

then

add and subtract to obtain the "Characteristics form" (single differential operator per equation)

$$\left.\begin{aligned}\left(\frac{\partial}{\partial s}+\tan\mu\,\frac{\partial}{\partial n}\right)(\omega+\vartheta)&=\frac{\tan\mu\sin\vartheta}{r}\\ \left(\frac{\partial}{\partial s}-\tan\mu\,\frac{\partial}{\partial n}\right)(\vartheta-\omega)&\frac{\tan\mu\sin\vartheta}{r}\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}\left(\cos\mu\frac{\partial}{\partial s}+\sin\mu\,\frac{\partial}{\partial n}\right)(\omega+\vartheta)&=\frac{\sin\mu\,\sin\vartheta}{r}\\ \left(\cos\mu\frac{\partial}{\partial s}-\sin\mu\,\frac{\partial}{\partial n}\right)(\vartheta-\omega)&=-\frac{\sin\mu\,\sin\vartheta}{r}\end{aligned}\right.$$

In (s, n) coordinates, $\vec{1}^{+}=\begin{Bmatrix}\cos\mu\\ \sin\mu\end{Bmatrix}$ $\qquad \vec{1}^{-}=\begin{Bmatrix}\cos\mu\\ -\sin\mu\end{Bmatrix}$



so $\left\{\begin{aligned}\cos\mu\frac{\partial}{\partial s}+\sin\mu\frac{\partial}{\partial n}&=\vec{1}^{+}\cdot\nabla=\frac{\partial}{\partial m^{+}}\\ \cos\mu\frac{\partial}{\partial s}-\sin\mu\frac{\partial}{\partial n}&=\vec{1}^{-}\cdot\nabla=\frac{\partial}{\partial m^{-}}\end{aligned}\right.$

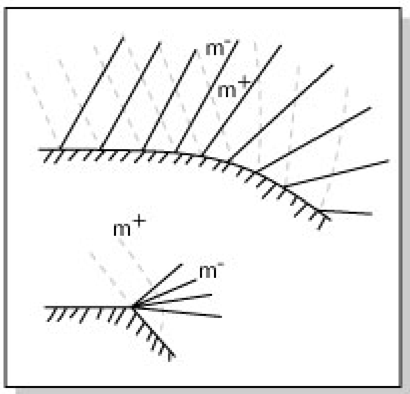
(m^+, m^- are <u>lengths</u> along characteristics)

$\begin{pmatrix} m^{-} \text{ inclined } \vartheta+\mu \\ m^{+} \text{ inclined } \vartheta-\mu\end{pmatrix}$

$$\begin{Bmatrix}\dfrac{\partial}{\partial m^{+}}(\vartheta+\omega)=+\dfrac{\sin\mu\,\sin\vartheta}{r}\\ \dfrac{\partial}{\partial m^{-}}(\vartheta-\omega)=-\dfrac{\sin\mu\,\sin\vartheta}{r}\end{Bmatrix}$$

For 2-D, $r\to\infty$ $\qquad\Rightarrow$ $\vartheta+\omega=const.$ along $m^{+}\equiv l^{+}$ (inclined $\vartheta-\mu$)
$\vartheta-\omega=const.$ along $m^{-}\equiv l^{-}$ (inclined $\vartheta+\mu$)

2-D Simple Regions



Consider a uniform region; the flow from it enters some disturbed region, like a wall turning. One of the m families originate in the uniform region (m^+ in example) and carries a constant invariant, e.g. $\vartheta + \omega = \vartheta_o + \omega_o$ everywhere downstream. Along one of the other characteristics (m^- here), we carry a constant (which varies from ch. To ch. of that family); we can evaluate it at the wall, for instance

$\vartheta - \omega = \vartheta_w - \omega_w$ along each m^- line and

$\omega_w = \vartheta_o + \omega_o - \vartheta_w$ so $\vartheta - \omega = 2\vartheta_w - (\vartheta_o + \omega_o)$

then
$$2\vartheta = \vartheta_o + \omega_o + \vartheta_w - \omega_w = 2\vartheta_w \text{ along each } m^- \rightarrow \quad \vartheta = \vartheta_w$$
$$2\omega = \vartheta_o + \omega_o - (\vartheta_w - \omega_w) \text{ along each } m^- \quad \rightarrow \quad \omega = \vartheta_o + \omega_o - \vartheta_w = \omega_w$$

so ϑ and ω (and hence M, μ) are constant along each m^-, and these m^- lines are straight (const. $\vartheta + \mu$).

This is a Simple Region (one of the invariants is constant). The turning of the flow is dictated by how ϑ changes on a m^+ line, as different m^- lines are crossed. Since on a m^+ we have $\vartheta + \omega = \vartheta_o + \omega_o$, changes of ϑ are equal and opposite to those in ω (ω increases as M does in the expansion, so ϑ decreases (increase negatively) at the same rate.) So, we can interpret ω as the magnitude of the isentropic flow turning in a simple region, i.e., when nothing varies along one characteristic family.

Calculation of ω (M)

$$d\omega = \sqrt{M^2 - 1}\frac{du}{u} = \sqrt{M^2 - 1}\left(\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}\right)$$

but $$c_p T + \frac{u^2}{2} = \text{const.} \Rightarrow T = \frac{T_0}{1 + \frac{\gamma - 1}{2} M^2} \Rightarrow \frac{dT}{T} = -\frac{\frac{\gamma - 1}{2} 2M\,dM}{1 + \frac{\gamma - 1}{2} M^2}$$

$$d\omega = \frac{\sqrt{M^2-1}}{M}\left(1-\frac{\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}\right)dM = \frac{\sqrt{M^2-1}}{M}\frac{dM}{1+\frac{\gamma-1}{2}M^2}$$

Integrates (with $\omega=0$ at $M=1$) to

$$\boxed{\omega = K\tan^{-1}\frac{\sqrt{M^2-1}}{K} - \tan^{-1}\sqrt{M^2-1}} \qquad \left(K=\sqrt{\frac{\gamma+1}{\gamma-1}}\right)$$

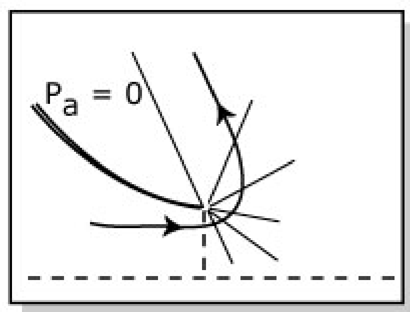
For $M\to\infty$ $\qquad \omega \to K\frac{\pi}{2}-\frac{\pi}{2} = \frac{\pi}{2}(K-1) = \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}}-1\right)$

γ	1.2	1.25	1.4	5/3
$\omega(M\to\infty)$	208°	180°	130.5°	90°

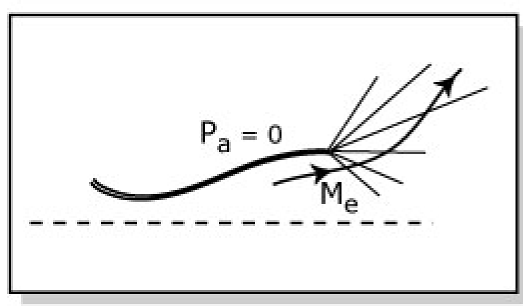
(Max. turning from M=1)

So rocket exhaust ($\gamma \approx 1.2-1.3$) can turn backwards at a sonic nozzle exit to vacuum)

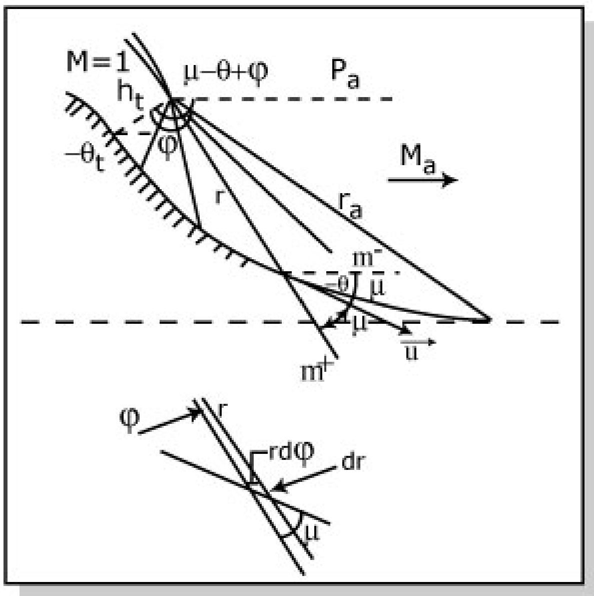
(but very long density, long mfp, so continuum approach fails at some point in the expansion, molecules then continue in straight line).



Actually, one starts from some high $M_e>1$, so the actual turning is through $\omega_\infty - \omega(M_e)$, not ω_∞, even in a vacuum:



Example of Application: Ideal 2-D plug Nozzle at Design Condition



Along m^-, $\vartheta - \omega = \vartheta_a - \omega_a$

Simple region (Prandtl-Meyer fan centered at lip L)

In particular, at inlet M=1, so $\boxed{\vartheta_{throat} = -\omega_a}$

$$\tan\mu = \frac{r\,d\varphi}{dr} = \frac{1}{\sqrt{M^2 - 1}}$$

From geometry, $\mu - \vartheta + \varphi = \frac{\pi}{2}\vartheta_t$

Sub. $\vartheta = \omega - \omega_a$ $\qquad\qquad \mu - \omega + \omega_a + \varphi = \frac{\pi}{2} + \omega_a$

$\vartheta_t = -\omega_a$ $\qquad\qquad \varphi = \frac{\pi}{2} + \omega - \mu$

$$\varphi = \frac{\pi}{2} + K\tan^{-1}\frac{\sqrt{M^2-1}}{K} \underbrace{- \tan^{-1}\sqrt{M^2-1} - \tan^{-1}\frac{1}{\sqrt{M^2-1}}}_{-\frac{\pi}{2}} \qquad \left(K = \sqrt{\frac{\gamma+1}{\gamma-1}}\right)$$

$$\varphi = K \tan^{-1} \frac{\sqrt{M^2 - 1}}{K}$$

$$\tan\frac{\varphi}{K} = \frac{\sqrt{M^2 - 1}}{K} = \frac{1}{K \tan\mu}$$

$$\Rightarrow \quad \frac{r\, d\varphi}{dr} = \frac{1}{K \tan\frac{\varphi}{K}} \qquad\qquad \frac{dr}{r} = K \frac{\sin\frac{\varphi}{K}}{\cos\frac{\varphi}{K}} d\varphi = -K^2 \frac{d\left(\cos\frac{\varphi}{K}\right)}{\cos\frac{\varphi}{K}}$$

$$r = \frac{\text{const.}}{\left(\cos\frac{\varphi}{K}\right)^{K^2}}$$

For $\varphi = 0, r = h_t$ (throat height)

$$r = \frac{h_t}{\left(\cos\frac{\varphi}{K}\right)^{K^2}}$$

In particular, at end of expansion $M = M_a \rightarrow \varphi_a = K \tan^{-1} \frac{\sqrt{M_a^2 - 1}}{K}$, then

$$r_a = \frac{h_t}{\left(\cos\frac{\varphi_a}{K}\right)^{K^2}}$$

and

$$h_a = r_a \sin\mu_a = \frac{r_a}{M_a}$$

$$x_a = r_a \cos\mu_a = r_a \sqrt{1 - \frac{1}{M_a^2}}$$

Numerical Application

$$\frac{P_o}{P_a} = 100 = \left(1 + \frac{\gamma - 1}{2} M_a^2\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow M_a = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{P_o}{P_a}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

Take $\gamma = 1.3$, $$M_a = \sqrt{\frac{2}{0.3}\left[100^{\frac{0.3}{1.3}} - 1\right]} = 3.554$$

Then $$\omega_a = K \tan^{-1}\left(\frac{M^2 - 1}{K}\right) - \tan^{-1}\sqrt{M^2 - 1} \qquad K = \sqrt{\frac{\gamma + 1}{\gamma - 1}} = \sqrt{\frac{2.3}{0.3}} = 2.769$$

$$\omega_a = 67.35^o \quad \Rightarrow \quad \vartheta_t = -67.35^o$$

Also $$\rho_a = K \tan^{-1}\left(\frac{M^2 - 1}{K}\right) = 141.01^o$$

and so $$\frac{r_a}{h_t} = \frac{1}{\left(\cos\frac{\rho_a}{K}\right)^{K^2}} = \frac{1}{\left(\cos\frac{141.01}{2.769}\right)^{2.769^2}}$$

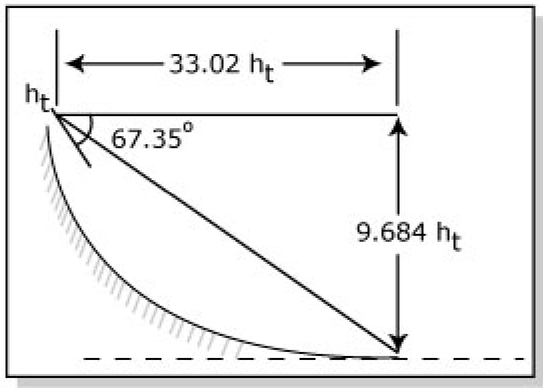
$$\boxed{\frac{r_a}{h_t} = 34.41}$$

From the geometry, $h_a = r_a \sin\mu_a = \frac{r_a}{M_a}$

$$\frac{h_a}{h_t} = \frac{1}{3.554}\ 34.41$$

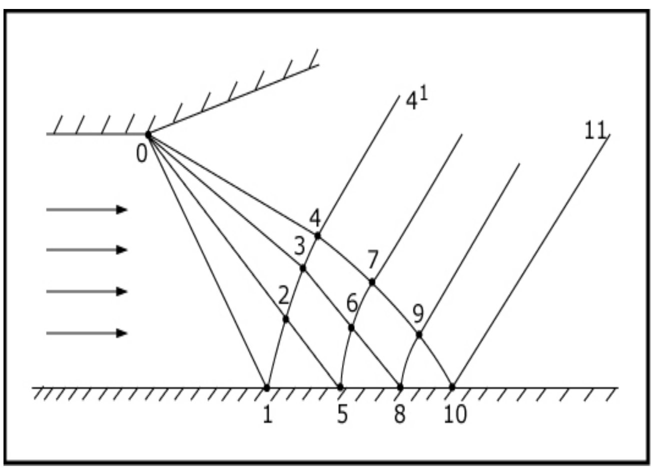
$$\boxed{\frac{h_a}{h_t} = 9.684}$$

and $\frac{x_a}{h_t} = r_a \cos\mu_a = 34.41\sqrt{1-\left(\frac{1}{3.554}\right)^2}$ $\boxed{\frac{x_a}{h_t} = 33.02}$



Very long and pointy, should be truncated.

Non-Simple Regions. When the characteristics of both families intersect some upstream disturbance, they affect each other's invariant, and characteristics are no longer straight, and no longer carry constant flow properties (except for their own invariant)



0-1-4 is a simple region

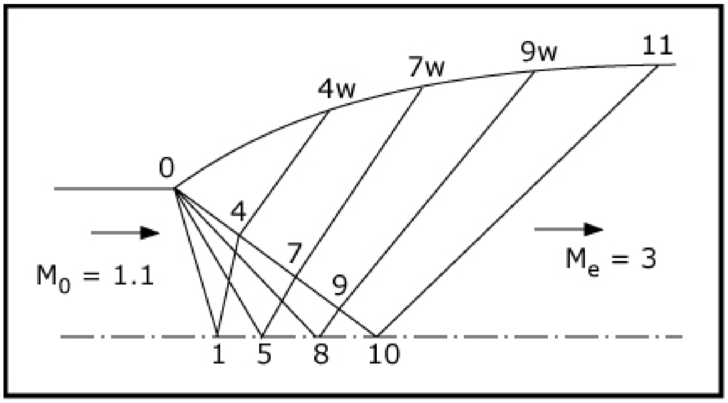
0-4-4^1 is uniform

1-4-10 is non-simple

4-4^1--11 is a simple region

Then we need to calculate in a step-by-step manner, carrying to each point the two invariants I^+, I^- from neighboring upstream points, along the m^+, m^- lines from them to us. After this is done, we know the new segments of m^+, m^- from our point (slopes $\vartheta+\mu$, $\vartheta-\mu$), so we can extend the grid as we go. Notice the flowfield properties can be found first everywhere and only then we need to come back and place the points geometrically.

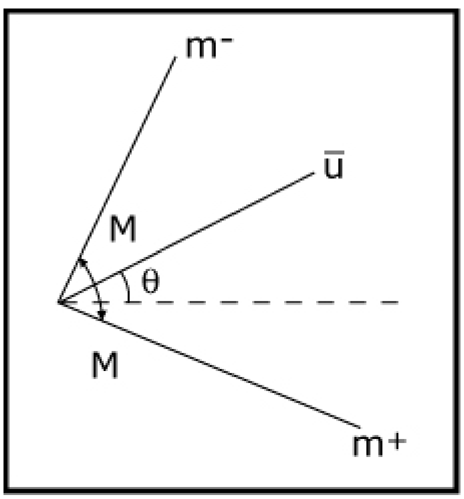
Example: Design a 2-D ideal nozzle to expand from near sonic conditions (M_0 = 1.1) to M_e = 3. Use only 4 characteristics. Use a corner expansion as a starter, $\gamma = 1.25$



$$K = \sqrt{\frac{2.25}{0.25}} = 3$$

$$\omega_o = 3\tan^{-1}\frac{\sqrt{1.21-1}}{3} - \tan^{-1}\sqrt{1.21-1} = 1.435^\circ \quad ; \; \vartheta_o = 0$$

$$\omega_e = 3\tan^{-1}\frac{\sqrt{9-1}}{3} - \tan^{-1}\sqrt{9-1} = 59.413^\circ \quad ; \; \vartheta_e = 0$$



At inlet: $I^- = \vartheta - \omega = \vartheta_o - \omega_o = -1.435^\circ$ (also at 4)

At exit: $I^+ = \vartheta + \omega = \vartheta_e + \omega_e = 59.413^\circ$ (also at 4, 7, 9, 10)

$$2\vartheta_4 = -1.435 + 59.413 = 57.978 \qquad \vartheta_4 = 28.989^\circ$$

At 4, then, $2\omega_4 = 59.413 + 1.435 = 60.848 \qquad \omega_4 = 30.424^\circ$

Select: $I_1^+ = 1.44^\circ$ $I_2^+ = 20.76^\circ$ $I_3^+ = 40.09^\circ$ $I_4^+ = 59.41^\circ$

Point	M	$I^+ = \vartheta + \omega\left(^\circ\right)$	$I^- = \vartheta - \omega\left(^\circ\right)$	$\omega\left(^\circ\right)$	$\vartheta\left(^\circ\right)$	$\mu\left(^\circ\right)$	$\vartheta + \mu\left(^\circ\right)$	$\vartheta - \mu\left(^\circ\right)$
1	1.1	+1.44	-1.44	1.44	0	65.38	65.38	-65.38°
2	1.439	20.76	-1.44	11.10	9.66	44.02	53.68	-34.36
3	1.726	40.09	-1.44	20.77	19.33	35.41	54.74	-16.08
4 (and 4w)	2.013	59.41	-1.44	30.43	28.99	29.79	58.78	-0.80
5	1.726	20.76	-20.76	20.76	0	35.41	35.41	-35.41
6	2.013	40.09	-20.76	30.43	9.67	29.79	39.46	-20.12
7 (and 7w)	2.315	59.41	-20.76	40.09	19.33	25.59	44.92	6.26
8	2.315	40.09	-40.09	40.09	0	25.59	25.59	-25.59
9 (and 9w)	2.641	59.41	-40.09	49.75	9.66	22.25	31.91	-12.59
10 (and 11)	.3	59.41	-59.41	59.41	0	19.47	19.47	-19.47

Notes

Points 1-4: same $\vartheta - \omega$

Point 5: Here $\vartheta = 0$ (a boundary condition) and $\vartheta + \omega$ = 20.76

Point 6-7: same $\vartheta - \omega$

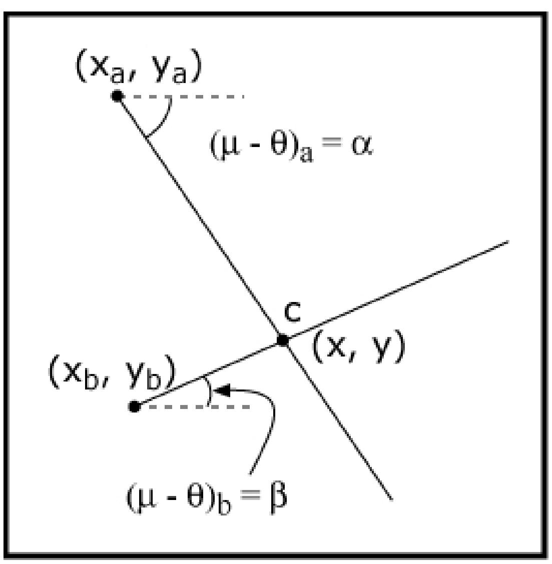
Point 8: Here $\vartheta = 0$ and $\vartheta + \omega$ = 40.09

Point 9: same $\vartheta - \omega$

Point 10: Here $\vartheta = 0$ and $\vartheta - \omega$ = 59.41

Note the very shallow angles of the m^+ lines (from 4, 7, 9, 10) which will put point 10 far to the right, and point 11 (at slope (m^+) of 19.47° even farther.

Locating the points geometrically



$$\left(x_c - x_a\right)\tan\alpha + \left(x_c - x_b\right)\tan\beta = y_a - y_b$$

$$x_c = \frac{y_a - y_b + x_a\tan\alpha + x_b\tan\beta}{\tan\alpha + \tan\beta}$$

$$\boxed{y_c - y_b = (x_c - x_b)\tan\beta} = \tan\beta \frac{(y_a - y_b) + (x_a - x_b)\tan\alpha}{\tan\alpha + \tan\beta}$$

$$y_c = \frac{y_b \tan\alpha + y_a \tan\beta + (x_a - x_b)\tan\alpha \tan\beta}{\tan\alpha + \tan\beta}$$

put x_a, y_a into 1, 2 $\tan\alpha$ into 3 x_b, y_b into 4.5 $\tan\beta$ into 6	Run 2 $\rightarrow$ x_c (st. in 7) Run 3 $\rightarrow$ y_c (st. in 7)

Finding the (x,y)'s is very laborious. Accuracy can be increased by averaging together the angles at the ends of each segment, which can be done because those angles come from the first pass. For instance, $\alpha = \frac{\alpha_a + \alpha_c}{2}$, $\overline{\beta} = \frac{\beta_b + \beta_c}{2}$.

Boundary with Prescribed Pressure

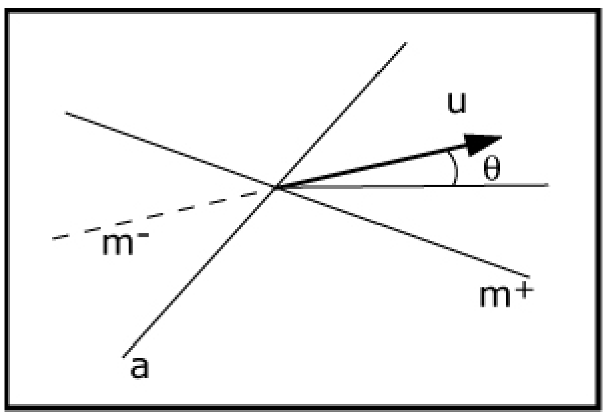
$$P = \frac{P_c}{\left(1+\frac{r-1}{2}M^2\right)^{r/r-1}} ,$$

and $M = M(w)$,

so $P = P(w)$ (given P_o)

or $w = w(P)$

So, if P is fixed on a boundary (contact surface), we can assign w there (just as we assign θ on a solid boundary)



From known point a,

$\theta - w = \theta_a - w_a$

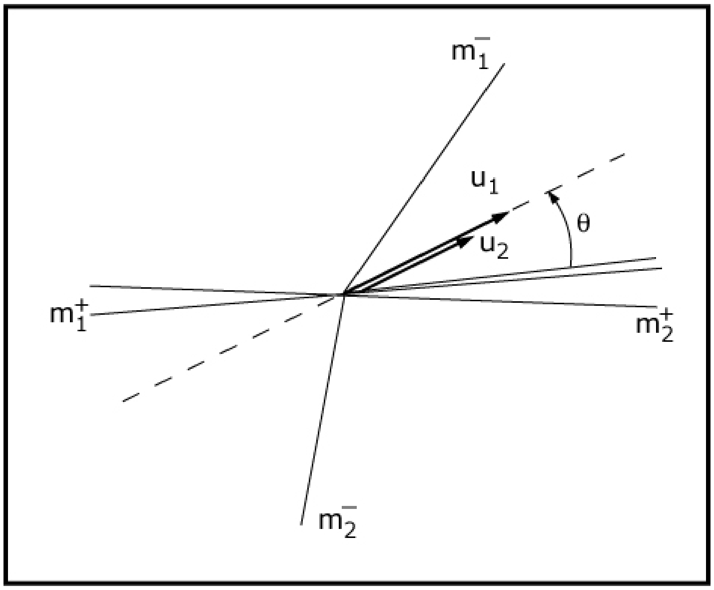
So $\theta = w(P) + \theta_a - w_a$

and this determines the slope of the boundary, and that of the "reflected" m^+

$\theta + w = 2w(P) + \theta_a - w_a$

More General Contact Surface Condition

If the outside fluid is also supersonic, we must solve on both sides of the contact surface, making sure P and θ are common at each boundary point



$P_1\left(w_1\right) = P_2\left(w_2\right)$

$\theta_1 = \theta_2$

From m_1^+ know $I_1^+ = \theta + w_1^+\left(P\right)$

From m_2^- know $I_2^- = \theta - w_2^-\left(P\right)$

So, $I_1^+ - I_2^- = w_1^+\left(P\right) + w_2^-\left(P\right)$

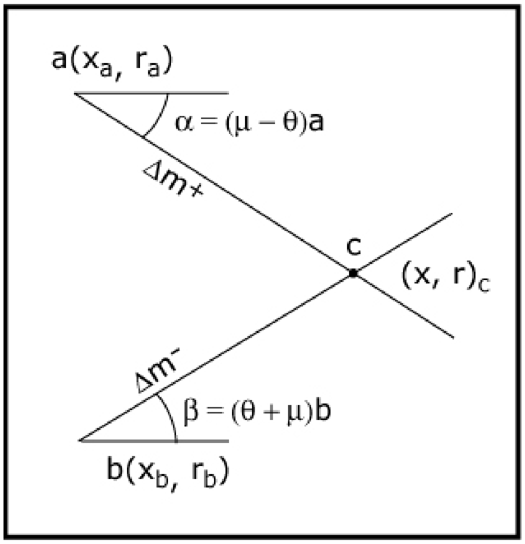
$\Rightarrow$ Solve for P $\Rightarrow w_1, w_2$

known now

and $\dfrac{I_1^+ + I_2^-}{2} = \theta + \dfrac{w_1^+\left(P\right) + w_2^-\left(P\right)}{2}$

<u>Modifications for Axisymmetric Conditions</u>

$$\left.\begin{aligned} \frac{\partial}{\partial m^+}\left(\theta + w\right) &= \frac{\sin\mu\sin\theta}{r} \\ \frac{\partial}{\partial m^-}\left(\theta - w\right) &= -\frac{\sin\mu\sin\theta}{r} \end{aligned}\right\}$$



(1) Calculate $(x,\ r)_c$ from the angle $(\mu-\theta)_a$, $(\mu+\theta)_b$

$$(x-x_a)\tan\alpha+(x-x_b)\tan\beta=r_a-r_b \Rightarrow x-\frac{x_a\tan\alpha+x_b\tan\beta+r_a-r_b}{\tan\alpha+\tan\beta}$$

and $r-r_b=(x-x_b)\tan\beta$

(2) $\Delta m^{+}=\dfrac{x-x_a}{\cos\alpha}$

$$\Delta m^{-}=\frac{x-x_b}{\cos\beta}$$

(3) Advance invariants $\theta+w$, $\theta-w$ based on μ,θ at a, b:

$$\left.\begin{aligned}\theta+w&=(\theta+w)_a+\frac{\sin\mu_a\ \sin\theta_a}{r_a}\Delta m^{+}\\ \theta-w&=(\theta-w)_b-\frac{\sin\mu_b\ \sin\theta_b}{r_b}\Delta m^{-}\end{aligned}\right\}$$

(r at c, computed in (1))

(4) $M=M(w)$, $\mu=\mu(M)$

(5) Iterate from (1) with α,β averaged between (a, b) and c:

$$\alpha\rightarrow\frac{(\mu-\theta)_a+(\mu-\theta)_c}{2}$$

$$\beta\rightarrow\frac{(\mu+\theta)_b+(\mu+\theta)_c}{2}$$

$$\left.\begin{aligned} r &\rightarrow \frac{r_a + r_b}{2} \\ \mu &\rightarrow \frac{\mu_a + \mu_b}{2} \\ \theta &\rightarrow \frac{\theta_a + \theta_b}{2} \end{aligned}\right\} \text{ better}$$

(6) Continue to new point.

So, computation of (θ, M) is now <u>coupled</u> to that of (x, r), whereas in 2-D (θ, M) can be found first. But the actual amount of computation is not much more (only the iteration stops).

On the axis, $r \rightarrow 0$, but $\theta = 0$, so

1. $x = x_a + r_a \tan\alpha$

2. $\Delta m^+ = \dfrac{r_a}{\sin\alpha}$

3. $\theta_c = 0$; $w_c = (\theta + w)_a + \dfrac{\sin\mu_a \sin\theta_a}{r_a}$ ← $r_a, \theta_a,$

4. $M_c = M(w_c)$, $\mu_c = \mu(M_c)$

5. $\alpha \rightarrow \dfrac{(\mu - \theta)_a + \mu_c}{2}$, go to (1) (once)

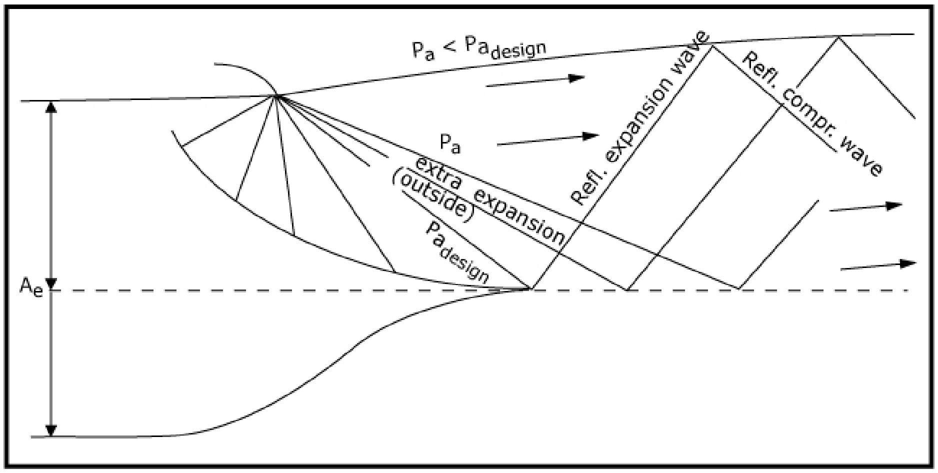
6. Continue

<u>Extension to Cases with Weak Shocks</u>

Since the entropy jump in a shock increases only as the <u>cube</u> of $\dfrac{\Delta P_{shock}}{\rho u_0^2}$, the isentropic assumption can be approximately extended when characteristics of our family show some mild convergence (in principle, that is always indicative of shock formation, because they carry conflicting information). When is the convergence too strong? Since characteristics are discretized, for weak convergence the ones of our family will converge, but <u>not cross</u>, and as long as they don't, it should be ⊔ OK. Of course, with finer resolution they <u>will cross</u>, but the loss of accuracy in ignoring that is of the same order as that increased by the coarse discretization in the first place.

This allows us to calculate off-design nozzle performance, like an overexpanded plug nozzle.

2-D Spike Nozzle with $P_a < P_{a_{design}}$



Pressure forces from hot-gas bathed surfaces are the same as at design. Net thrust is increased because the P_a contribution P_aA_e is reduced:

$$F = F_{des.} + \left(P_{a_{des.}} - P_a\right) A_e$$

and so $F_{vac} = F_{des.} + P_{a_{des.}} A_e$ (just as for a bell)

"Practical" Nozzle Designs

Ideal nozzle are too long, last portion has small wall angle, so small thrust contribution. With small **?,** maybe negative contribution. So, options:

(a) Constrain length, ask for contour that gives highest thrust/given L. Methods of calculus of variations (Raw nozzle, Ref : " Exhaust Nozzle Contour from Optimum Thrust", Jet Propulsion 28 (June 1958): 377-382. Exit flow non-parallel, non-uniform, computationally high.

(b) Ad-hoc method (widely used) is to truncate contract an ideal nozzle.

(1) Design wall contour for desired P_e/P_c
(2) If longer than desired L truncate it to some intermediate area ratio.
(3) Contract this truncated nozzle to desired length

$$x'_a = x'_b \frac{x_c}{x_d} = x'_b \frac{L_{desired}}{L_{ideal}}$$

(4) Translate profile to right metal kink at P smoothes out.

Gives “adequate” performance but less than **?** nozzle. Not really justified. (Hoffman, J. D. J. of Propulsion 3 (March-April 1987): 150-156.

XRS-2200 Engine Data

Thrust, lbf	
At Sea Level	206,500
In Vacuum	268,000
Specific Impulse, sec.	
At Sea Level	339
In Vacuum	439
Propellants	O2, H2
Mixture Ratio (O/F)	5.5
Chamber Pressure, psia	857
Cycle	Gas Generator
Area Ratio	58
Throttling, Percent Thrust	40-119
Differential Throttling	+/- 15%
Dimensions, Inches	
Forward End	133 high x 88 wide
Aft End	46 high x 88 wide
Forward to Aft	79