PROFESSOR: In the twelfth lecture, we're going to talk about maxima and minima. Let's finish up what we did last time. We really only just started with maxima and minima. And then we're going to talk about related rates. So, right now I want to give you some examples of max-min problems. And we're going to start with a fairly basic one. So what's the thing about max-min problems? The main thing is that we're asking you to do a little bit more of the interpretation of word problems. So many of the problems are expressed in terms of words. And so, in this case, we have a wire which is length 1. Cut into two pieces. And then each piece encloses a square. Sorry, encloses a square. And the problem - so this is the setup. And the problem is to find the largest area enclosed.

So here's the problem. Now, in all of these cases, in all these cases, there's a bunch of words. And your job is typically to draw a diagram. So the first thing you want to do is to draw a diagram. In this case, it can be fairly schematic. Here's your unit length. And when you draw the diagram, you're going to have to pick variables. So those are really the two main tasks. To set up the problem. So you're drawing a diagram. This is like word problems of old, in grade school through high school. Draw a diagram and name the variables. So we'll be doing a lot of that today. So here's my unit length. And I'm going to choose the variable x to be the length of one of the pieces of wire. And that makes the other piece 1 - x. And that's pretty much the whole diagram, except that there's something that we did with the wire after we cut it in half. Namely, we built two little boxes out of it. Like this, these are our squares. And their side lengths are x/4 and (1-x)/4. So, so far, so good.

And now we have to think, well, we want to find the largest area. So I need a formula for area in terms of variables that I've described. And so that's the last thing. I'll give the letter A as the label for the area. And then the area is just the square of x/4 plus the square of 1 minus x - whoops, that strange 2 got in here - over 4. So far, so good. Now, the instinct that you'll have, and I'm going to yield to that instinct, is we should charge ahead and just differentiate. All right? That's all right. We'll find the critical points. So we know that those are important points. So we're going to find the critical points. In other words, we take the derivative, we set the

derivative of A with respect to x = 0.

So if I do that differentiation, I get the, well, so the first one, x^2 / 16, that's 8. Sorry. That's x/8, right? That's the derivative of this. And if I differentiate this, I get well, the derivative of 1 - x^2 is 2(1-x)(a-1). So it's -(1-x)/8. So there are two minus signs in there, I'll let you ponder that differentiation, which I did by the chain rule. Hang on a sec, OK? Just wait until we're done. So here's the derivative. Is there a problem?

STUDENT: [INAUDIBLE]

PROFESSOR: Right, so there's a 1/16 here. This is x^2 / 16. And so it's 2x over 8, over 16, sorry. Which has an 8. That's OK. All right, so now, This is equal to 0 if and only if x = 1 - x. That's 2x = 1, or in other words x = 1/2. All right? So there's our critical point. So x = 1/2 is the critical point. And the critical value, which is what you get when you evaluate A at 1/2, is (1/2) / 4, that's 1/8. So that's (1/8)^2 + (1/8)^2 which is 1/32. So, so far, so good. But we're not done yet. We're not done.

So why aren't we done? Because we haven't checked the end points. So let's check the end points. Now, in this problem, the end points are really sort of excluded. The ends are between 0 and 1 here. That's the possible lengths of the cut. And so what we should really be doing is evaluating in the limit, so that would be the right-hand limit as x goes to 0 of A. And if you plug in x = 0, what you get here is 0 + (1/4)^2, which is 1/16. And similarly, at the other end, that's 1-, from the left, we get (1/4)^2 + 0, which is also 1/16. So, what you see is that the schematic picture of this function, and this isn't even so far off from being the right picture here, is that its level here is 1/16 and then it dips down and goes up. Right? This is 1/2, this is 1, and this level here is a half that. This is 1/32.

So we did not find, when we found the critical point we did not find the largest area enclosed. We found the least area enclosed. So if you don't pay attention to what the function looks like, not only will you about half the time get the wrong answer, you'll get the absolute worst answer. You'll get the one which is the polar opposite from what you want. So you have to pay a little bit of attention to the function that you've got. And in this case it's just very schematic. It dips down and goes up, and that's true of pretty much most functions. They're fairly simple. They maybe only have one critical point. They only turn around once. But then, maybe the critical point is the maximum or maybe it's the minimum. Or maybe it's neither, in fact. So we'll be discussing that maybe some other time.

So what we find here is that we have the least area enclosed is 1/32. And this is true when x = 1/2. So these are equal squares. And most when there's only one square. Which is more or less the limiting situation, if one of the pieces disappears. Now, so that's the first kind of example. And I just want to make one more comment about terminology before we go on. And I will introduce it with the following question. What is the minimum? So, what is the minimum? Yeah.

STUDENT: [INAUDIBLE]

PROFESSOR: Right. The lowest value of the function. So the answer to that question is 1/32. Now, the problem with this question and you will-- so that refers to the minimum value. But then there's this other question which is where is the minimum. And the answer to that is x = 1/2. So one of them is the minimum point, and the other one is the minimum value. So they're two separate things. Now, the problem is that people are sloppy. And especially since you usually find the critical point first, and the value, that is plugging in for A, second, people will stop short and they'll give the wrong answer to the question, for instance.

Now, both questions are important to answer. You just need to have a word to put there. So this is a little bit careless. When we say what is the minimum, some people will say 1/2. And that's literally wrong. They know what they mean. But it's just wrong. And when people ask this question, they're being sloppy. Anyway. They should maybe be a little clearer and say what's the minimum value. Or, where is the value achieved. It's achieved at, or where is the minimum value achieved. "Where is min achieved?", would be a better way of phrasing this second question. So that it has an unambiguous answer.

And when people ask you for the minimum point, they're also-- so why is it that we call it the minimum point? We have this word, critical point, which is what x = 1/2 is, and critical value. And so I'm making those same distinctions here. But there's another notion of a minimum point, and this is an alternative if you like. The minimum point is the point (1/2, 1/32). Right, that's a point on the graph. It's the point - well, so that graph is way up there, but I'll just put it on there. That's this point. And you might say min there. And you might point to this point, and you might say max. And similarly, this one might be a max. So in other words, what this means is simply that people are a little sloppy. And sometimes they mean one thing and sometimes they mean another. And you're just stuck with this, because there'll be some authors who will say one thing and some people will mean another and you just have to live with this little bit of annoying ambiguity. Yeah?

STUDENT: [INAUDIBLE]

PROFESSOR: OK, so that's a good - very good. So here we go, find the largest area enclosed. So that's sort of a trick question, isn't it? So there are various - that's a good thing to ask. That's sort of a trick question, why? Because according to the rules, we're trapped between the two maxima at something which is strictly below. So in other words, one answer to this question would be, and this is the answer that I would probably give, is 1/16. But that's not really true. Because that's only in the limit. As x goes to 0, or as x goes to 1-. And if you like, the most is when you've only got one square. Which breaks the rules of the problem. So, essentially, it's a trick question. But I would answer it this way. Because that's the most interesting part of the answer, which is that it's 1/16 and it occurs really when one of the squares disappears to nothing.

So now, let's do another example here. And I just want to illustrate the second style, or the second type of question. Yeah.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, since the question was, what was the largest area, why did we find the least area. The reason is that when we go about our procedure of looking for the least, or the most, we'll automatically find both. Because we don't know which one is which until we compare values. And actually, it's much more to your advantage to figure out both the maximum and minimum whenever you answer such a question. Because otherwise you won't understand the behavior of the function very well. So, the question. We started out with one question, we answered both. We answered two questions. We answered the question of what the largest and the smallest value was.

STUDENT: Also, I'm wondering if you can check both the minimum [INAUDIBLE] approaches [INAUDIBLE].

PROFESSOR: Yes. One can also use-- the question is, can we use the second derivative test. And the answer is, yes we can. In fact, you can actually also stare at this and see that it's a sum of squares. So it's always curving up. It's a parabola with a positive second coefficient. So you can differentiate this twice. If you do you'll get 1/8 plus another 1/8 and you'll get 1/16. So the second derivative is 1/16. Is 1/4. And that's an acceptable way of figuring it out. I'll mention the second derivative test again, in this second example. So let me talk about a second example.

So again, this is going to be another question.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, when I say minimum or maximum point which will I mean.

STUDENT: [INAUDIBLE]

PROFESSOR: So I just repeated the question. So the question is, when I say minimum point, what will I mean? OK? And the answer is that for the purposes of this class I will probably avoid saying that. But I will say, probably, where is the minimum achieved. Just in order to avoid that. If I actually said it, I often am referring to the graph, and I mean this. And in fact, when you get your little review for the second exam, I'll say exactly that on the review sheet. And I'll make this very clear when we were doing this. However, I just want to prepare you for the fact that in real life, and even me when I'm talking colloquially, when I say what's the minimum point of something, I might actually be mixing it up with this other notion here.

So let's do another example. So this is an example to get us used to the notion of constraints. So we have, so consider a box without a top. Or, if you like, we're going to find the box without a top. With least surface area for a fixed volume. Find the box without a top with least surface area for a fixed volume. The procedure for working this out is the following. You make this diagram. And you set up the variables. In this case, we're going to have four names of variables. We have four letters that we have to choose. And we'll choose them in a kind of a standard way, alright? So first I have to tell you one more thing. Which is something that we could calculate separately but I'm just going to give it to you in advance. Which is that it turns out that the best box has a square bottom. And that's going to get rid of one of our variables for us. So it's got a square bottom, and so let's draw a picture of it. So here's our box. Well, that goes down like this, almost. Maybe I should get it a little farther down.

So here's our box. Let's correct that just a bit. So now, what about the dimensions of this box? Well, this is going to be x, and this is very foreshortened, but it's also x. The bottom is x by x, it's the same dimensions. And then the vertical dimension is y. So far, so good. Now, I promised you two more letter names. I want to compute the volume. The volume is, the base is x^2, and the height is y. So there's the volume. And then the area, the area is the area of the bottom, which is x^2, that's the bottom. And then there are the four sides. And the four sides are rectangles of dimensions x, y. So it's 4xy. So these are the sides. And remember, there's no top. So that's our setup.

So now, the difference between this problem and the last problem is that there are two variables floating around, namely x and y, which are not determined. But there's what's called a constraint here. Namely, we've fixed the relationship between x and y. And so, that means that we can solve for y in terms of x. So $y = V/x^2$. And then, we can plug that into the formula for A. So here we have A which is $x^2 + 4x v/x^2$. Question.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, will you need to know this intuitively? No. That's something that I would have to give to you. I mean, it's actually true that a lot of things, the correct answer is something symmetric. In this last problem, the minimum turned out to be exactly halfway in between because there were sort of equal demands from the two sides. And similarly, here, what happens is if you elongate one side, you get less - it actually is involved with a two variable problem. Namely, if you have a rectangle and you have a certain amount of length associated with it, what's the optimal thing you can do with that. But I won't-- in other words, the optimal rectangle, the least perimeter rectangle, turns out to be a square. That's the little sub-problem that leads you to this square bottom. But so that would have been a separate max-min problem. Which I'm skipping, because I want to do this slightly more interesting one.

So now, here's our formula for A, and now I want to follow the same procedure as before. Namely, we look for the critical point. Or points. So let's take a look. So again, A is $(x^2 + 4v) / x$. And A' is $2x - 4v/x^2$. So if we set that equal to 0, we get $2x = 2v/x^2$. So $2x^3$. How did that happen to change into 2? Interesting, guess that's wrong. OK. So this is $x^3 2v$. And so $x = 2^{(1/3)} v^{(1/3)}$. So this is the critical point. So we are not done. Right? We're not done, because we don't even know whether this is going to give us the worst box or the best box, from this point of view. The one that uses the most surface area or the least. So let's check the ends, right away. To see what's happening. So can somebody tell me what the ends, what the end values of x are? Where does x range from?

STUDENT: [INAUDIBLE]

PROFESSOR: What's the smallest x can be, yeah.

STUDENT: [INAUDIBLE]

PROFESSOR: OK, the claim was that the largest x could be root A, because somehow there's this x^2 here

and you can't get any further past than that. But there's a key feature here of this problem. Which is that A is variable. The only thing that's fixed in the problem is V. So if V is fixed, what do you know about x?

STUDENT: [INAUDIBLE]

PROFESSOR: $x > 0$, yeah. The lower end point, that's safe. Because that has to do geometrically with the fact that we don't have any boxes with negative dimensions. That would be refused by the Post Office, definitely. Over and above the empty top, which they wouldn't accept either.

STUDENT: [INAUDIBLE]

PROFESSOR: It's true that x is less than square root of V / y. So that's using this relationship. But notice that $y = V/x^2$. So 0 to infinity, I just got a guess there over here, that's right. Here's the upper limit. So this is really important to realize. This is most problems. Most problems, the variable, if it doesn't have a limitation, usually just goes out to infinity. And infinity is a very important end for the problem. It's usually an easy end to the problem, too. So there's a possibility that if we push all the way down to $x = 0$, we'll get a better box. It would be very strange box. A little bit like our vanishing enclosure. And maybe an infinitely long box, also very inconvenient one. Might be the best box. We'll have to see.

So let's just take a look at what happens. So we're looking at A, at 0+. And that's $x^2 + 4V / x$ with x at 0+. So what happens to that? Notice right here, this is going to infinity. So this is infinite. So that turns out to be a bad box. Let's take a look at the other end. So this is $x^2 + 4V / x$, x going to infinity. And again, this term here means that this thing is infinite. So the shape of this thing, I'll draw this tiny little schematic diagram over here. The shape of this thing is like this, right? And so, when we find that one turnaround point, which happened to be at this strange point 2/3-- sorry, $2^{(1/3)} V^{(1/3)}$, that is going to be the minimum. So we've just discovered that it's the minimum. Which is just what we were hoping for. This is going to be the optimal box.

Now, since you asked earlier and since it's worth checking this as well, let's also check an alternative justification. So an alternative to checking ends is the second derivative test. I do not recommend the second derivative test. I try my best, when I give you problems, to make it really hard to apply the second derivative test. But in this example, the function is simple enough so that it's perfectly OK. If you take the derivative here, remember, this was whatever it was, $2x - 4V / x^2$. If I take the second derivative, it's $2 + 8V / x^3$. And that's positive. So this

thing is concave up. And that's consistent with its being-- the critical point is a min. Is a minimum point. See how I almost said, is a min, as opposed to minimum point. So watch out. Yes.

STUDENT: [INAUDIBLE]

PROFESSOR: You're one step ahead of me. The question is, is this the answer to the question or would we have to give y and A and so on and so forth. So, again, this is something that I want to emphasize and take my time with right now. Because it depends, what kind of real-life problem you're answering, what kind of answer is appropriate. So, so far we've found the critical point. We haven't found the critical value. We haven't found the dimensions of the box. So we're going to spend a little bit more time on this, exactly in order to address these questions.

So, first of all. The value of y. So, so far we have $x = 2^{1/3} V^{1/3}$. And certainly if you're going to build the box, you also want to know what the y-value is. The y-value is going to be, let's see. Well, it's V / x^2, so that's $V / (2^{1/3} V^{1/3})^2$, which comes out to be $2^{2/3} V^{1/3}$. So there's the y-value. On top of that, we could figure out the value of A. So that's also a perfectly reasonable part of the answer. Depending on what one is interested in, you might care how much money it's going to cost you to build this box. This optimal box. And so you plug in the value of A. So A, let's see, is up here. It's $x x^2 + 4V/x$.

So that's going to be $(2^{1/3} V^{1/3})^2$ plus $4V / (2^{1/3} V^{1/3})$. And if you work that all out, what you get turns out to be $3 * 2^{1/3} V^{2/3}$.

So if you like, one way of answering this question is these three things. That would be the minimum point corresponding to the graph. That would be the answer to this question. But the reason why I'm carrying it out in such detail is I want to show you that there are much more meaningful ways of answering this question. So let me explain that. So let me go through some more meaningful answers here.

The first more meaningful answer is the following idea. Simply, what are known as dimensionless variables. So the first thing that you notice is the scaling law. That $A / V^{2/3}$ is the thing that's a dimensionless quantity. That happens to be $3 * 2^{1/3}$. So that's one thing. If you want to expand the volume, you'll have to expand the area by the 2/3 power of the volume. And if you think of the area as being in, say, square inches, and the volume of the box as being in cubic inches, then you can see that this is a dimensionless quantity and you have a dimensionless number here, which is a characteristic independent of what A and V were.

The other dimensionless quantity is the ratio of x to y. Or x to y. So, again, that's inches divided by inches. And it's $2^{1/3} V^{1/3}$ divided by $2^{-2/3} V^{1/3}$, which happens to be 2. So this is actually the best answer to the question. And it shows you that the box is a 2:1 box. If this is 2 and this is 1, that's the good box. And that is just the shape, if you like, and it's the optimal shape. And certainly that, aesthetically, that's the cleanest answer to the question. There was a question right here. Yes.

STUDENT: [INAUDIBLE]

PROFESSOR: Could you repeat that, I couldn't hear.

STUDENT: I'm wondering if you'd be able to get that answer if you [INAUDIBLE] square.

PROFESSOR: The question is, could we have gotten the answer if we weren't told that the bottom was square. The answer is, yes in 18.02 with multivariable. You would have to have three letters here, an x, a y, and a z, if you like. And then you'd have to work with all three of them. So I separated out into one, there's a separate one variable problem that you can do for the base. And then this is a second one variable problem for this other thing. And it's just two consecutive one variable problems that solve the multivariable problem. Or, as I say in multivariable calculus, you can just do it all in one step. Yeah?

STUDENT: [INAUDIBLE]

PROFESSOR: Why did I divide x by y, rather than y by x, or in any? So, again, what I was aiming for was dimensionless quantities. So x and y are measured in the same units. And also the proportions of the box - so that's another word for this is proportions - are something that's universal, independent of the volume V. It's something you can say about any box, at any scale. Whether it be, you know, something by Cristo in the Common. Maybe we'll get in here to do some fancy--

STUDENT: [INAUDIBLE]

PROFESSOR: The proportions is - with geometric problems typically, when there's a scaling to the problem, where the answer is the same at small scales and at large scales - this is capturing that. So that's why, the ratios are what's capturing that. And that's why it's aesthetically the nicest thing to ask.

STUDENT: So, what exactly does the ratio of the area to the volume ratio [INAUDIBLE] tell us?

PROFESSOR: Unfortunately, this number is a really obscure number. So the question is what does this tell us. The only thing that I want to emphasize is what's on the left-hand side here. Which is, it's the area to the 2/3 power over the volume, so it's a dimensionless quantity that happens to be this. If you do this, for example, in general with planar diagrams, circumference to area is a bad ratio to take. What you want to take is the square of circumference to area. Because the square of circumference has the same dimensions, that is, say, inches squared to area. Which is in square inches. So, again, it's these dimensionless quantities that you want to cook up. And those are the ones that will have universal properties. The most famous of these is the circle that encloses the most area for its circumference. And, again, that's only true if you take the square of the circumference. You do the units correctly.

Anyway. So we're here, we've got a shape. We've got an answer to this question. And I now want to do this problem. Well, let's put it this way. I wanted to do this problem by a different method. I think I'll take the time to do it. So I want to do this problem by a slightly different method here. So, here's Example 2 by implicit differentiation. So the same example, but now I'm going to do it by implicit differentiation. Well, I'll tell you the advantages and disadvantages to this method here. So the situation is, you have to start the same way. So here is the starting place of the problem. And the goal was the minimum of A with V constant. So this was the situation that we were in. And now, what I want to do is just differentiate. The function y is implicitly a function of x, so I can differentiate the first expression. And that yields $0 = 2xy + x^2 y'$. So this is giving me my implicit formula for y', So $y' = -2xy / x^2$. Or in other words, $-2y/x$.

And then I also have the dA/dx. Now, you may notice I'm not using primes quite as much. Because all of the variables are varying, and so here I'm emphasizing that it's a differentiation with respect to the variable x. And this becomes $2x + 4y + 4xy'$. So again, this is using the product rule. And now I can plug in for what y' is, which is right above it. So this is $2x + 4y + 4x(-2y/x)$. And that's equal to 0. And so let's gather that together.

So what do we have? We have $2x + 4y$, and then, altogether, this is $8 - 8y$, equals 0. So that's the same thing as $2x = 4y$. The $- 4y$ goes to the other side. And so, $x/y = 2$. So this, I claim, so you have to decide for yourself. But I claim that this is faster. It's faster, and also it gets to the heart of the matter, which is this scale-invariant proportions. Which is basically also nicer. So it gets to the nicer answer, also. So those are the advantages that this has. So it's faster, and it gets to this, I'm going to call it nicer. And the disadvantage is it did not check whether this

critical point is a max, min, or neither. So we didn't quite finish the problem. But we got to the answer very fast. Yeah, question.

STUDENT: [INAUDIBLE]

PROFESSOR: How would you check it?

STUDENT: [INAUDIBLE]

PROFESSOR: Well, so it gives you a candidate. The answer is-- so the question is, how would you check it? The answer is that for this particular problem, the only way to do it is to do something like this. So in other words, it doesn't save you that much time. But with many, many, examples, you actually can tell immediately that if the two ends, the thing is, say, 0, and inside it's positive. Things like that. So in many, many, cases this is just as good.

So now I'm going to change subjects here. But the subject that I'm going to talk about next is almost-- is very, very closely linked. Namely, I talked about implicit differentiation. Now, we're going to just talk about dealing with lots of variables. And rates of change. So, essentially, we're going to talk about the same type of thing. So, I'm going to tell you about a subject which is called related rates. Which is really just another excuse for getting used to setting up variables and equations. So, here we go. Related rates. And I'm going to illustrate this with one example today, one tomorrow. So here's my example for today. So, again, this is going to be a police problem. But this is going to be a word problem and-- sorry, I'm don't want to scare you, no police. Well, there are police in the story but they're not present. So, but I'm going to draw it immediately with the diagram because I'm going to save us the trouble. Although, you know, the point here is to get from the words to the diagram.

So you have the police, and they're 30 feet from the road. And here's the road. And you're coming along, here, in your, let's see, in your car going in this direction here. And the police have radar. Which is bouncing off of your car. And what they read off is that you're 50 feet away. They also know that you're approaching along the line of the radar at a rate of 80 feet per second. Now, the question is, are you speeding. That's the question. So when you're speeding, by the way, 95 feet per second is about 65 miles per hour. So that's the threshold here. So what I want to do now is show you how you set up a problem like this.

This distance is 50. This is 30, and because it's the distance to a straight line you know that this is a right angle. So we know that this is a right triangle. And this is set out to be a right

triangle, which is an easy one, a 3, 4, 5 right triangle just so that we can do the computations easily. So now, the question is, how do we put the letters in to make this problem work, to figure out what the rate of change is. So now, let me explain that right now. And we will actually do the computation next time.

So the first thing is, you have to understand what's changing and what's not. And we're going to use t for time, in seconds. And now, an important distance here is the distance to this foot of this perpendicular. So I'm going to name that x. I'm going to give that letter x. Now, x is varying. The reason why I need a letter for it as opposed to this 40 is that it's going to have a rate of change with respect to t. And, in fact, it's related to-- the question is whether dx/dt is faster or slower than 95. So that's the thing that's varying. Now, there's something else that's varying.

This distance here is also varying. So we need a letter for that. We do not need a letter for this. Because it's never changing. We're assuming the police are parked. They're not ready to roar out and catch you just yet, and they're certainly not in motion when they've got the radar guns aimed at you. So you need to know something about the sociology and style of police. So you need to know things about the real world.

Now, the last bit is, what about this 80 here. So this is how fast you're approaching. Now, that's measured along the radar gun. I claim that that's d/dt of this quantity here. So this is D is also changing. That's why we needed a letter for it, too. So, next time, we'll just put that all together and compute dx / dt.