

Derivative of $\sin x$, Algebraic Proof

A specific derivative formula tells us how to take the derivative of a specific function: if $f(x) = x^n$ then $f'(x) = nx^{n-1}$. We'll now compute a specific formula for the derivative of the function $\sin x$.

As before, we begin with the definition of the derivative:

$$\frac{d}{dx} \sin x = \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

You may remember the following angle sum formula from high school:

$$\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$$

This lets us untangle the x from the Δx as follows:

$$\frac{d}{dx} \sin x = \lim_{\Delta x \rightarrow 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin(x)}{\Delta x}$$

We can simplify this expression using some basic algebraic facts:

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{\Delta x \rightarrow 0} \left[\frac{\sin x \cos \Delta x - \sin x}{\Delta x} + \frac{\cos x \sin \Delta x}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{\sin x (\cos \Delta x - 1)}{\Delta x} + \frac{\cos x \sin \Delta x}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\sin x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) + \cos x \left(\frac{\sin \Delta x}{\Delta x} \right) \right] \\ \frac{d}{dx} \sin x &= \lim_{\Delta x \rightarrow 0} \sin x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) + \lim_{\Delta x \rightarrow 0} \cos x \left(\frac{\sin \Delta x}{\Delta x} \right) \end{aligned}$$

We now have two familiar functions – $\sin x$ and $\cos x$ – and two ugly looking fractions to deal with. The fractions may be familiar from our discussion of removable discontinuities.

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{\cos \Delta x - 1}{\Delta x} &= 0 \\ \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} &= 1. \end{aligned}$$

Using these (as yet unproven) facts,

$$\lim_{\Delta x \rightarrow 0} \sin x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) + \lim_{\Delta x \rightarrow 0} \cos x \left(\frac{\sin \Delta x}{\Delta x} \right)$$

simplifies to $\sin x \cdot 0 + \cos x \cdot 1 = \cos x$. We conclude:

$$\frac{d}{dx} \sin x = \cos x$$

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