Example: $\lim_{x\to 0} \dfrac{\sin x}{x^2}$

If we apply l'Hôpital's rule to this problem we get:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x^2} &= \lim_{x\to 0}\frac{\cos x}{2x} \qquad \text{(l'Hop)}\\ &= \lim_{x\to 0}\frac{-\sin x}{2} \qquad \text{(l'Hop)}\\ &= 0.\end{aligned}$$

If we instead apply the linear approximation method and plug in $\sin x \approx x$, we get:

$$\begin{aligned}\frac{\sin x}{x^2} &\approx \frac{x}{x^2}\\ &\approx \frac{1}{x}.\end{aligned}$$

We then conclude that:

$$\begin{aligned}\lim_{x\to 0^+}\frac{\sin x}{x^2} &= \infty\\ \lim_{x\to 0^-}\frac{\sin x}{x^2} &= -\infty.\end{aligned}$$

There's something fishy going on here. What's wrong?

Student: L'Hôpital's rule wasn't applied correctly the second time.

That's correct; $\lim_{x\to 0}\frac{\cos x}{2x}$ is of the form $\frac{1}{0}$, not $\frac{0}{0}$ or some other indeterminate form.

This is where you have to be careful when using l'Hôpital's rule. You have to verify that you have an indeterminate form like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying the rule. The moral of the story is: **Look before you l'Hôp**.

Also, don't use l'Hospital's rule as a crutch. If we want to evaluate:

$$\lim_{x\to\infty}\frac{x^5-2x^4+1}{x^4+2}$$

we can apply l'Hôpital's rule four times, or we could divide the numerator and denominator by x^5 to conclude:

$$\begin{aligned}\lim_{x\to\infty}\frac{x^5-2x^4+1}{x^4+2} &= \lim_{x\to\infty}\frac{1-2/x+1/x^5}{1/x+2/x^5}\\ &= \frac{1}{0}\\ &= \infty.\end{aligned}$$

After enough practice with rates of growth, we can calculate this limit almost instantly:

$$\lim_{x\to\infty}\frac{x^5-2x^4+1}{x^4+2} \sim \lim_{x\to\infty}\frac{x^5}{x^4} = \infty.$$