$\lim_{x\to\infty}(x^{1/x})$

Use an extension of l'Hôpital's rule to compute $\lim_{x\to\infty}(x^{1/x})$.

**Solution**

This calculation is very similar to the calculation of $\lim_{x\to 0^+} x^x$ presented in lecture, except that instead of the indeterminate form $0^0$ we instead have $\infty^0$.

As before, we use the exponential and natural log functions to rephrase the problem:

$$x^{1/x} = e^{\ln x^{1/x}} = e^{\frac{\ln x}{x}}.$$

Thus, $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}}$. Since the function $e^t$ is continuous,

$$\lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^{\lim_{x\to\infty}\frac{\ln x}{x}}.$$

We can now focus our attention on the limit in the exponent; $\lim_{x\to\infty}\dfrac{\ln x}{x}$ is in the indeterminate form $\frac{\infty}{\infty}$, so l'Hôpital's rule is applicable.

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln x}{x} &= \lim_{x\to\infty}\frac{1/x}{1} \qquad \text{(provided the limit exists)}\\ &= \frac{0}{1}\\ &= 0\end{aligned}$$

We conclude that $\lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty}\frac{\ln x}{x}} = 1$.

This implies that the $n^{\text{th}}$ root of $n$ approaches 1 as $n$ approaches infinity.