

PROFESSOR: Welcome to this recitation on the phase portrait. So you're given matrix A , a two-by-two matrix with entries $\begin{pmatrix} -1 & 1 \\ 1 & -1-c \end{pmatrix}$, where c is a constant we will be varying, and -1 . As c increases from 0 to other positive values, determine the path of the system on the trace-determinant diagram that we saw in previous recitations and draw the phase portraits that corresponds to the critical points and the nature of the critical points.

So here, we're basically talking of a system that would be a two-by-two system where we have derivatives of entries x, y for vector, equals to this matrix, A_c , multiplying the unknown vector x_c . So take a few minutes and work out through this problem and I'll be right back.

Welcome back. So I already prepared the trace-determinant diagram for you. So here as a reminder, this is the parabola that determines whether we're going to have repeated eigenvalues or not. Above this parabola, we have two complex conjugate eigenvalues. Below this parabola, we have real eigenvalues, either of the same signs, positive or negative, or of different signs. And then, we have the borderline cases.

In our case here, we're looking in a matrix A with the trace equals to $-1 - 1$, so it's -2 , and a determinant that is equal to $2 - c$. So here, we are basically along this dotted line where the trace equals to -2 . So either we're going to have complex values with negative real parts or negative eigenvalues. So we're in stable configurations.

So we're going to be moving at c equal to 0 from a case where-- I'm just going to label this point 1. For example, we have the matrix, $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ with the trace equals to -2 , and the determinant equals to 2. So clearly, we are this point. And we are just going to be increasing the value of c and moving along these lines, crossing this first boundary case where we will have either defective or not case. And I'll discuss very briefly what we have.

And then, we have other value where we are basically in this area where we have real eigenvalues, both of them negative, so we have stability. And it would basically be a sink. And then we cross this other borderline case where the determinant is equal to 0. And here, we have one eigenvalue that is equal to 0 and one that is negative. And then, we get to do the part of the diagram where the determinant is negative, and the trace is negative. And we basically have a saddle structure where we have one eigenvalue as negative and another eigenvalue that is positive.

So let's get through these five cases. And I'll just do more detailed picture illustrating what I just said. So just write down a bit more in detail the first case here. So we have the case where the eigenvalues are both complex conjugate, and we have their real part being negative. So basically, we have spirals, asymptotically stable spirals. And the spirals could be either clockwise or counterclockwise. And one way of determining the direction of these spirals is to look at the sign of the velocity vector of the trajectories in the phase space. So the phase space basically diagram here where we have x and y as the axes of this space.

So let's look, for example, at a particular point here that would be velocity vector $[1, 0]$. And this position vector $[1, 0]$ would give us a velocity vector of minus 1 and plus 1. So basically, this vector would be directed upward, which means that we are in the case where we would have a spiral coming this way toward the critical point and basically with the velocity vector here going this way.

You don't have to actually do all this to figure out which direction of the spiral you should choose. You can just look-- for these cases, two-by-two matrices-- at the entry at the lower left part of your matrix. And if the coefficient here is positive, then it will determine the direction of the velocity vector at this point. So if it's positive in this case, you would have this counterclockwise direction of rotation of the spiral.

So now we can move on and do the following cases a bit faster. So the second case, we are in the case where we now have two repeated eigenvalues. Both of them are negative. So we can have either a complete case. If we had a matrix that was diagonal, we would have basically a star. But that's not the case, because if we pick, for example, the value c equals to 1 that puts us on the parabola, we can see that the matrix A_c is not diagonal. So we are in a defective case.

The defective case, if you compute the eigenvalues and eigenvector would give you something that looks like that. The first ray of the eigenvector corresponding to your eigenvalues would be in the direction of $[1, 0]$. We have negative eigenvalues, so it's going to 0. And basically, to determine the direction, you have to see that we are also following a transition here on the trace-determinant diagram. And so basically, we will have a spiral that would look like this. What am I doing? Am I doing the same thing?

And it would basically rotate, as you can see here, in the same direction as the case that we had just before. Because we basically have the transition of the structure of the critical point.

And the arrows are pointing toward the critical point again, because our two eigenvalues are basically here negative. It is just a repeated eigenvalue that is negative.

So now we can move on to a third case where here we move into the wedge area where we have the two eigenvalues now being real, and both of them are negative. So for example, we would have two eigenvalues that would give us two eigenvectors that are basically, they're rays. The direction of the trajectories on these two rays would be toward the critical point, because we have, again, negative eigenvalues. And the trajectories then would be following the lambda. So if I, for example, pick λ_1 less than λ_2 , then I would have something like this going to 0. And we would have this trajectory corresponding to the eigenvalue that is closer to 0. And again, the arrows would be going toward this critical point.

So we can now move on. And if we keep increasing c , we reach now the point 4 where we are in the special case, again, a boundary case where the determinant is 0. But the trace is non-zero. If the determinant is 0, which is the product of the two eigenvalues, it means that we have one eigenvalue that is equal to 0 and another eigenvalue that is actually, in this case, minus 2, so just negative.

So what happens here? What happens is that we have now the eigenvector that corresponds to the eigenvalue equals to 0 is basically just defining a whole line of critical points. So all the points on this line that would correspond, so for example, to $\lambda = 0$ are all critical points. So we don't need to actually draw arrows here. All the points would be critical points.

And another direction or ray would be determined by the second eigenvalue, λ_2 . And we would have, for example, directions in this way. It would correspond to the direction of v_2 . So all these would be parallel to v_2 . And we would have the trajectories going toward all the critical points on this line. So in this case, we're in a case where we actually don't have one localized critical point like in the previous case that we saw. We've actually have a whole line of critical points.

So let's move one more point down. So here, another value of c , we now are in the lower part of the diagram where we now have two eigenvalues that are again real. But now, the determinant is negative. Determinant is the product of the two. So basically we know that one is positive and the other one is negative.

So for example here, we would have one positive, the negative eigenvalue. So we would have the positive eigenvalue and the negative eigenvalue here. So here, again, we're just moving in

the structure of the critical point. So we should have a smooth transition between the diagrams going 1 to 5.

And so here, what we see is that we actually have the ray that is basically the stable space here that would give us the trajectories going to 0. But it's the only region where we have stability. And all the trajectories will just then be tangent in minus infinity to the ray v_1 , corresponding to the negative eigenvalue, and going towards the direction of the ray with a positive eigenvalue at plus infinity.

So we would have something like that, that would correspond to a saddle. This would be a comb. This is basically a sink or basically a stable node. This would be defective stable node. And here, we just have a asymptotically stable spiral.

So basically here what you could see is that we have, by just changing the value of the constant c , had moved the structure of the critical point of the system from the phase diagrams one to five. And the transitions can be seen to be smooth if you had a continuum of values for c . So that ends this recitation.