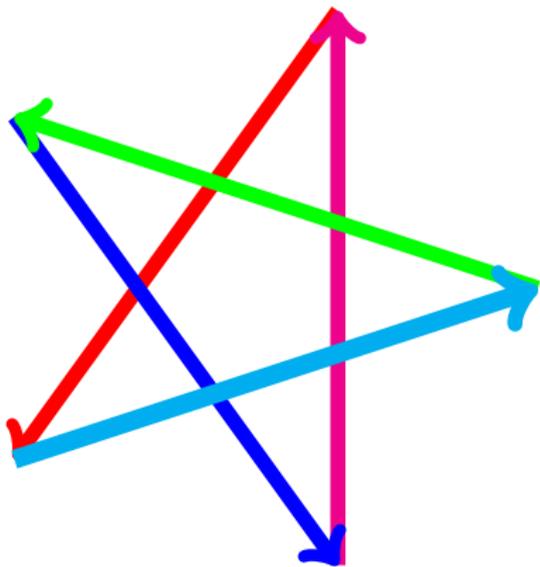


Studio 2
18.05 Spring 2014



Expected Value

If X is a random variable that takes values x_1, x_2, \dots, x_n then the *expected value* of X is defined by

$$E(X) = p(x_1)x_1 + p(x_2)x_2 + \dots + p(x_n)x_n = \sum_{i=1}^n p(x_i) x_i$$

- Weighted average
- Measure of central tendency

Properties of $E(X)$

1. $E(X + Y) = E(X) + E(Y)$
2. $E(aX + b) = aE(X) + b$
3. $E(h(X)) = \sum_i h(x_i) p(x_i)$

Examples

Example 1. Find $E(X)$

1. $X:$ 3 4 5 6

2. pmf: 1/4 1/2 1/8 1/8

3. $E(X) = 3/4 + 4/2 + 5/8 + 6/8 = 33/8$

Example 2. Suppose $X \sim \text{Bernoulli}(p)$. Find $E(X)$.

1. $X:$ 0 1

2. pmf: 1 - p p

3. $E(X) = (1 - p) \cdot 0 + p \cdot 1 = p.$

Example 3. Suppose $X \sim \text{Binomial}(12, .25)$. Find $E(X)$.

$X = X_1 + X_2 + \dots + X_{12}$, where $X_i \sim \text{Bernoulli}(.25)$. Therefore

$$E(X) = E(X_1) + E(X_2) + \dots + E(X_{12}) = 12 \cdot (.25) = 3$$

In general if $X \sim \text{Binomial}(n, p)$ then $E(X) = np$.

Board Question

Suppose (hypothetically!) that everyone at your table gets up, does a board question, and sits back down at random (i.e., all seating arrangements are equally likely).

What is the expected number of people who return to their original seat?

The solution is on the next slides. There is more in the slides for class 4.

Solution

Number the people from 1 to n . Let X_i be the Bernoulli random variable with value 1 if person i returns to their original seat and value 0 otherwise. Since person i is equally likely to sit back down in any of the n seats, the probability that person i returns to their original seat is $1/n$. Therefore $X_i \sim \text{Bernoulli}(1/n)$ and $E(X_i) = 1/n$. Let X be the number of people sitting in their original seat after the rearrangement. Then

$$X = X_1 + X_2 + \cdots + X_n.$$

By linearity of expected values, we have

$$E(X) = \sum_{i=1}^n E(X_i) = \sum_{i=1}^n 1/n = 1.$$

- It's neat that the expected value is 1 for any n .
- If $n = 2$, then both people either retain their seats or exchange seats. So $P(X = 0) = 1/2$ and $P(X = 2) = 1/2$. In this case, X never equals $E(X)$.
- The X_i are not independent (e.g. for $n = 2$, $X_1 = 1$ implies $X_2 = 1$).
- Expectation behaves linearly even when the variables are dependent.

R Exercises

Suppose $Y \sim \text{Binomial}(8, .6)$.

1. Run a simulation with 1000 trials to estimate $P(Y = 6)$ and $P(Y \leq 6)$
2. Use R and the formula for binomial probabilities to compute $P(Y=6)$ exactly.

answer: 1. Here's code that will run the simulations

```
ntosses = 8
ntrials = 5000
phead = .6
x = rbinom(ntrials * ntosses, 1, phead)
trials = matrix(x, nrow = ntosses, ncol = ntrials)
y = colSums(trials)
mean(y == 6)
mean(y <= 6)
```

2. Here's code that will do the computation

```
choose(ntosses, 6) * phead^6 * (1-phead)^(ntosses-6)
```

R Exercises

3. A friend has a coin with probability .6 of heads. She proposes the following gambling game.

- You will toss it 10 times and count the number of heads.
- The amount you win or lose on k heads is given by $k^2 - 7k$

(a) Plot the payoff function.

(b) Make an exact computation using R to decide if this is a good bet.

(c) Run a simulation and see that it approximates your computation in part (b).

answer: Code is given on the next slides. Code is also posted with the transcript and a detailed description of the solution with code is posted with the studio 2 materials.

Solution

```
# (a) Plot the payoff as a function of  $k$ 
outcomes = 0:10
payoff = outcomes^2 - 7*outcomes
plot(outcomes, payoff, pch=19) # plot for part (a)

# (b) Compute  $E(Y)$ 
phead = .6
ntosses = 10
outcomes = 0:ntosses
payoff = outcomes^2 - 7*outcomes

# Compute the entire vector of probabilities using dbinom
countProbabilities = dbinom(outcomes, ntosses, phead)
expectedValue = sum(countProbabilities*payoff) # This is
the weighted sum
expectedValue # part (b)
```

Solution continued

```
# (c) Simulation
```

```
phead = .6
```

```
ntosses = 10
```

```
ntrials = 1000
```

```
# We use rbinom to generate a vector of ntrials binomial  
outcomes
```

```
trials = rbinom(ntrials, ntosses, phead)
```

```
# trials is a vector of counts. We can apply the payoff  
formula to the entire vector
```

```
payoffs = trials^2 - 7*trials
```

```
mean(payoffs)
```

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18.05 Introduction to Probability and Statistics

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