18.100B Fall 2010
Practice Quiz 4 Solutions

1.

$$\bullet L(f, P = (x_i)) = \sum_{i=1}^{n} \underbrace{\inf_{x \in [x_{i-1}, x_i]} f(x)}_{\geq 0} \underbrace{(x_i - x_{i-1})}_{>0} \geq 0$$

$$\Rightarrow L(f) = \sup_P L(f, P) \geq 0$$

$$\bullet U(f, P = (x_i)) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$$

Given $\epsilon > 0$, pick a partition
$P^\epsilon = (0, \frac{1}{2^n} - \delta, \frac{1}{2^n} + \delta, \frac{3}{2^n} - \delta, \frac{3}{2^n} + \delta, \ldots, \frac{2^n-1}{2^n} - \delta, \frac{2^n-1}{2^n} + \delta, 1)$
with $\delta < \frac{1}{2^n}$ (hence $\frac{1}{2^n} + \delta < \frac{3}{2^n} - \delta$) and $\delta \leq \epsilon$, then

$$U(f, P^\epsilon) = \underbrace{\sum_{k=1}^{2^n-1} \underbrace{\sup f}_{\frac{1}{2^n}} \cdot \underbrace{\Delta x_i}_{2\delta}}_{\text{from intervals } [\frac{k}{2^n}-\delta, \frac{k}{2^n}+\delta]} + \underbrace{0}_{\text{from other intervals}} = \frac{2^n-1}{2^{n-1}}\delta < 2\delta < \epsilon$$

Together with Rudin ($L(f) \leq U(f)$) this shows
$0 \leq L(f) \leq U(f) \leq \epsilon \ \forall \epsilon > 0 \Rightarrow L(f) = U(f) = 0$
$\Rightarrow f$ integrable, $\int_0^1 f dx = L(f) = 0$

2. f continuous $\Rightarrow$ integrable $\Rightarrow \sup_P L(f, P) = L(f) = 0$
$\Rightarrow \forall$ partitions $P, L(f, P) \leq 0$
Suppose by contradiction $f(x_0) > 0$ for some $x_0 \in [a, b]$,
then by continuity find $\delta > 0$ s.t.
$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{1}{2} f(x_0)$
$\Rightarrow f(x) \geq f(x_0) - \frac{1}{2} f(x_0) = \frac{1}{2} f(x_0)$
Now consider any equidistant partition P with $\Delta x < \delta$, then

$$L(f, P) = \sum_{i=1}^{n} \underbrace{\inf_{x \in [x_{i-1}, x_i]} f(x)}_{\geq 0 \text{ since } f(x) \geq 0} \Delta x \underbrace{\geq}_{\text{just looking at an interval that contains } x_0} \inf_{x \in [x_{i_0-1}, x_{i_0}]} f(x) \cdot \Delta x$$

$\geq \frac{1}{2} f(x_0) \cdot \Delta x > 0$
in contradiction to $L(f, P) \leq 0$

3.(a) $m > n$

$$||f_n - f_m||_\infty = \left|\left| \sum_{k=n+1}^{m} e^{-kx} \cos(kx) \right|\right|_\infty \leq \sum_{k=n+1}^{m} \sup_{x \geq a} e^{-kx} |\cos kx| \leq \sum_{k=n+1}^{m} e^{-ka} \underset{n \to \infty}{\longrightarrow} 0$$

since $\sum e^{-ka} = \sum (e^{-a})^k$ converges due to $|e^{-a}| < 1$
This shows uniform convergence by the "Cauchy criterion" (in Rudin).

(b) To show that f is continuous at $x_0 \in (0, \infty)$, note that
• each f_n is continuous on $[\frac{x_0}{2}, 2x_0]$
• $f_n \to f$ uniformly on $[\frac{x_0}{2}, 2x_0]$ by (a)
So, by Rudin, f is continuous on $[\frac{x_0}{2}, 2x_0]$, which contains x_0.

(c)
$$\int_1^\infty f(x)dx \underset{\text{by definition}}{=} \lim_{b\to\infty} \int_1^b f(x)dx \underset{\text{by Rudin}}{=} \lim_{b\to\infty} \lim_{n\to\infty} \int_1^b f_n(x)dx$$

$\displaystyle \lim_{n\to\infty} \int_1^b f_n(x)dx \underset{\text{linearity of integral}}{=} \lim_{n\to\infty} \sum_{k=1} \int_1^b \underbrace{e^{-kx} \cos kx}_{|\cdot| \le e^{-kx}} dx$ exists by comparison with the absolutely convergent series $\displaystyle\sum_{k=1}^\infty e^{-k}$, and

$$\left| \int_1^b f(x)dx \right| = \left| \lim_{n\to\infty} \int_1^b f_n(x)dx \right| \le \sum_{k=1}^n \int_1^b e^{-kx}dx = \sum_{k=1}^n \left[\tfrac{-1}{k} e^{-kx}\right]_1^b \le \sum_{k=1}^n \tfrac{1}{k} e^{-k}$$
$$\le \sum_{k=1}^n \left(\frac{1}{e}\right)^k \le \sum_{k=1}^\infty \left(\frac{1}{e}\right)^k = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$$

Similarly, $\displaystyle\lim_{b\to\infty} \int_1^b f dx$ exists since for $b' \ge b$

$$\left| \int_1^{b'} f dx - \int_1^b f dx \right| = \lim_{n\to\infty} \left| \int_b^{b'} f_n dx \right| \le \sum_{k=1}^\infty \tfrac{1}{k}(e^{-bk} - e^{-b'k}) \le \sum_{k=1}^\infty (e^{-b})^k = \tfrac{1}{1-e^{-b}}$$

converges to 0 as $b \to \infty$. (Hence the same holds for any sequence $b_i \to \infty$, making $\int_1^{b_i} f dx$ a Cauchy sequence. Completeness of $\mathbb{R}$ then implies convergence as $i \to \infty$; and the limit for all sequences $b_i \to \infty$ is the same since otherwise one could make a divergence (oscillating) sequence.)

So $\displaystyle\int_1^\infty f dx$ exists, and $\displaystyle\int_1^\infty f dx = \lim_{b\to\infty} \int_1^b f dx \le \lim_{b\to\infty} \frac{e}{e-1} = \frac{e}{e-1}$.

4.(a) FALSE
Differentiable implies continuous, but not bounded - e.g. $f(x) = x^{-1}$ on $[0, 1]$ is differentiable on (0,1).

(b) TRUE
See Rudin.

(c) TRUE
$$L(f, P) = \sum \underbrace{\inf_{x\in[x_{i-1},x_i]} f(x)}_{\le 0 \text{ since any interval contains } x\in\mathbb{R}\setminus\mathbb{Q}} \cdot \Delta x_i \le 0 \Rightarrow L(f) = \int f dx \le 0$$

(d) TRUE
The limit is continuous by Rudin 7...., and uniformly continuous since [a,b] is compact

(e) TRUE
Almost everywhere continuous $\iff$ Riemann integrable
So result follows from "$f_n \in \mathbb{R}, ||f_n - f||_\infty \to 0 \Rightarrow f \in \mathbb{R}$"

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18.100B Analysis I
Fall 2010

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