18.100B/C: Fall 2010
Solutions to Practice Final Exam

1. Suppose for sake of contradiction that $x > 0$. Then $\frac{1}{2} \cdot x > 0$ because the product of two positive quantities is positive. Thus $\frac{x}{2} + 0 < \frac{x}{2} + \frac{x}{2}$ (because $y < z$ implies $x + y < x + z$ for all x), i.e., $\frac{x}{2} < x$. Also, since $\varepsilon := \frac{x}{2} > 0$ we have by assumption that $x \leq \frac{x}{2}$. However, for a strict order at most one of $x < \frac{x}{2}$ and $\frac{x}{2} < x$ can be true. Hence we obtain a contradiction to the assumption $x > 0$. Thus $x \not> 0$. Since $x \geq 0$, this implies $x = 0$, as desired.

2.(a) We use $2xy \leq x^2 + y^2$ (which follows from $(x-y)^2 \geq 0$) and $a_n \geq 0$ to estimate

$$0 < \sqrt{a_n a_{n+1}} \leq \frac{1}{2}(\sqrt{a_n}^2 + \sqrt{a_{n+1}}^2) = \frac{1}{2}a_n + \frac{1}{2}a_{n+1}.$$

Next, the partial sums of $\sum_{n=1}^{\infty} a_{n+1}$ are the same ones (shifted by one – see (b)) as for $\sum_{n=1}^{\infty} a_n$, and so by assumption both series converge. Hence by linearity for limits, the series $\sum_{n=1}^{\infty} \frac{1}{2}a_n + \frac{1}{2}a_{n+1}$ also converges. Now convergence of $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ follows from the comparison criterion.
(b) Since $a_{n+1} \leq a_n$ we obtain $0 \leq a_{n+1} \leq \sqrt{a_n a_{n+1}}$, so $\sum_{n=1}^{\infty} a_{n+1}$ converges by the comparison test. But now $\lim_{k\to\infty} \sum_{n=1}^{k} a_{n+1}$ exists iff $\lim_{k\to\infty} \sum_{n=0}^{k-1} a_{n+1} = \lim_{k\to\infty} \sum_{n=1}^{k} a_n$ exists; proving convergence of the latter.

3.(a) Both $f(x) = 4x(1-x)$ and $f(x) = 1 - |2x-1|$ work nicely.
(b) No function: continuous functions take connected sets to connected sets.
(c) Define

$$f(x) = \begin{cases} 0, & x \leq 1, \\ 1, & x \geq 2. \end{cases}$$

This function is continuous on $[0,1] \cup [2,3]$ and $f([0,1] \cup [2,3]) = \{0,1\}$.
(d) No function: suppose such a function f exists. There exists x_1 for which $f(x_1) = 1$ and x_2 for which $f(x_2) = 2$, so by the Intermediate Value Theorem there is x between x_1 and x_2 for which $f(x) = \sqrt{2}$, a contradiction. (Or, use connectedness again.)
(e) No function: continuous functions take compact sets to compact sets.

4. Given $\varepsilon > 0$, by uniform convergence of (f_n), we can choose some $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$. By uniform continuity of f_N, we can choose some δ such that $d(x,y) < \delta$ implies $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$. Then for any $x, y \in E$ such that $d(x,y) < \delta$ we have

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

so f is uniformly continuous.

5. (see Melrose Test 2)

6. (see Melrose Test 2)

7.(a) For all $\varepsilon > 0$, there exists a (countable) collection $\{B(x_i, r_i)\}$ of open balls such that $N \subset \bigcup_i B(x_i, r_i)$ and $\sum_i r_i < \varepsilon$.
(b) We have $\{x \mid f(x) \neq f(x)\} = \emptyset$ has measure 0, so $f \sim f$. The relation is symmetric since $\{x \mid g(x) \neq f(x)\} = \{x \mid g(x) \neq f(x)\}$. To check is transitivity assume $f \sim g$ and $g \sim h$. Observe that if $f(x) \neq h(x)$ then we must have either $f(x) \neq g(x)$ or $g(x) \neq h(x)$ (or both), so

$$\{x \mid f(x) \neq h(x)\} \subseteq \{x \mid f(x) \neq g(x)\} \cup \{x \mid f(x) \neq g(x)\}.$$

So we must show that unions and subsets of measure-0 sets have measure 0. For subsets, just take a covering of the superset of measure 0 to cover its subset. For the union, take the union of two coverings of measure less than $\varepsilon/2$ to cover the union with sets of total measure less than ε.
(c) Since f and g are both integrable, $f - g$ is integrable as well, and we are asked to show that $\int_0^1 f - g = 0$ given that $f - g = 0$ almost everywhere. Since $f - g$ is integrable, the integral is equal to the infimum over all upper Riemann sums. Since $f - g$ is zero almost everywhere, every interval contains a point at which $f - g = 0$, so the upper Riemann sum for any fixed partition is a sum of nonnegative numbers and thus nonnegative. The infimum of a set of nonnegative quantities must itself be nonnegative, so $\int_0^1 f - g \geq 0$. However, we may apply identical reasoning to get that $\int_0^1 g - f \geq 0$. Since these two quantities are negatives of each other, they both must equal 0, as needed.

8.(a) Fix $\varepsilon > 0$. For each f_i, choose a δ_i such that $d(x, y) < \delta_i$ implies $|f_i(x) - f_i(y)| < \varepsilon$ for all x, y. Then let $\delta = \min\{\delta_i\} > 0$ and we have that for any $f_i \in \mathscr{F}$ and any x, y in the common domain that if $d(x, y) < \delta$ then $|f_i(x) - f_i(y)| < \varepsilon$, so $\mathscr{F}$ is equicontinuous.
(b) Let $\delta = \varepsilon/n$. If $|x - y| < \delta$ then

$$|f_n(x) - f_n(y)| = \left| \frac{x}{x + \frac{1}{n}} - \frac{y}{y + \frac{1}{n}} \right| = \frac{\frac{|x-y|}{n}}{\left|x + \frac{1}{n}\right| \cdot \left|y + \frac{1}{n}\right|} \leq \frac{\frac{|x-y|}{n}}{\frac{1}{n^2}} = n|x - y| < \varepsilon,$$

so f_n is uniformly continuous for all n.
(c) We have $f_n(0) = 0$ for all n and $f_n(x) \to 1$ as $n \to \infty$ for any fixed $x \in (0, 1]$, so (f_n) converges pointwise to the function

$$f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \in (0, 1]. \end{cases}$$

However, $f_n(\frac{1}{n}) = \frac{1}{2}$ for all n, so for all n there exists x such that $d(f_n(x), f(x)) > \frac{1}{3}$. Thus no subsequence of the (f_n) can converge uniformly. (Alternatively, invoke problem 4 here: if convergence were uniform, the limit function would be uniformly continuous, when in fact it's not even continuous.) In addition, we have $0 \leq f_n(x) \leq 1$ for all $n \in \mathbb{N}$ and all $x \in [0, 1]$, so (f_n) is uniformly bounded. By Arzelà-Ascoli, any equicontinuous pointwise bounded sequence of continuous functions has a uniformly convergent subsequence, so it follows that our sequence of functions is not equicontinuous.

9. see Melrose Test 1 .. hence no solution here

10.(a) Choose some f such that $f(x_0) = c \neq 0$. Then if $d_\infty(f, g) < \frac{|c|}{2}$, it follows that

$$\begin{aligned}
|g(x_0)| &= |g(x_0) - f(x_0) + f(x_0)| \\
&\geq |f(x_0)| - |f(x_0) - g(x_0)| \\
&\geq |f(x_0)| - \sup_{x \in X} |f(x) - g(x)| \\
&> |c| - \frac{|c|}{2} \\
&> 0,
\end{aligned}$$

so $g(x_0) \neq 0$. Thus there is an open ball in K_0^C around every element of K_0^C, so K_0^C is open and thus K_0 is closed.

(b) Denote the set of the previous part by $K_0(x)$. Then

$$K_1 = \bigcap_{x \in E} K_0(x)$$

is an intersection of closed sets, and so closed. We showed on one of the problem sets that if two continuous functions agree on a dense subset of a metric space then they agree on the whole space, so it follows that actually $K_1 = \{z\}$ where z is the function such that $z(x) = 0$ for all x.

(c) We have that actually $B = B_1(z)$ is the open ball of radius 1 centered at the all-zero function z, and we've shown that an open ball in any metric space is an open set. (As a reminder of this general result: choose any $x \in B_r(z)$, and let $d = d(x, z) < r$. Then for any $y \in B_{r-d}(x)$ we have $d(z, y) \leq d(z, x) + d(x, y) < d + (r - d) = r$, so $y \in B_r(z)$, as needed.)

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