18.100B/C Practice Final Exam

Closed book, no calculators.

YOUR NAME: ______________________________

This is a 180-minute exam. No notes, books, or calculators are permitted. Point values (out of 100) are indicated for each problem. There is a (hard) bonus question, Problem 9, at the end – do not attempt it until you have worked all other problems. (Note, you can achieve the full 100 points without attempting the bonus problem.) Do all the work on these pages.

Problem 1. [10 points] Suppose that $x \in \mathbb{R}$ satisfies $0 \leq x \leq \epsilon$ for every $\epsilon > 0$. Show that $x = 0$, using only axioms of $\mathbb{R}$ as an ordered field. State the axioms you are using. (Note that the Archimedean and least upper bound properties are *not* ordered field axioms.)

Problem 2. [5+5 points] Let (a_n) be a sequence of positive real numbers.

(a) Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ also converges.

(b) Show that the converse is also true if (a_n) is monotone decreasing: i.e. in this case, if $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges then so does $\sum_{n=1}^{\infty} a_n$.

Problem 3. [10 points: each part /2] For each of the following examples, either give an example of a continuous function f on S such that $f(S) = T$, or explain why there can be no such continuous function.

(a) $S = (0, 1)$, $T = (0, 1]$.

(b) $S = (0, 1)$, $T = (0, 1) \cup (1, 2)$.

(c) $S = [0, 1] \cup [2, 3]$, $T = \{0, 1\}$.

(d) $S = \mathbb{R}$, $T = \mathbb{Q}$.

(e) $S = [0, 1] \times [0, 1]$, $T = (0, 1) \times (0, 1)$.

Problem 4. [10 points] Assume $f_n\colon E \to \mathbb{R}$, $E \subseteq \mathbb{R}$ are uniformly continuous functions. Assume f_n converges to f uniformly. Prove that f is also uniformly continuous.

Problem 5. [10 points] Let $f : X \longrightarrow Y$ be a continuous map between metric spaces and let $K \subset X$ be compact. Prove that $f(K) \subset Y$ is compact using the definition of compactness through open covers.

Problem 6. [10 points] Let $\alpha : [0,1] \longrightarrow \mathbb{R}$ be given by

$$\alpha(x) = \begin{cases} x - 1 & 0 \leq x < \frac{1}{2} \\ x + 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

and let $f(x) = 2x$. Explain why the Riemann-Stieltjes integral $\int_0^1 f d\alpha$ exists and compute its value, justifying your arguments carefully.

Problem 7. [5+5+5 points]

(a) Let N be a subset of $\mathbb{R}$. What does it mean to say "N has measure 0"? State the precise definition.

(b) Let $\mathscr{F}(\mathbb{R})$ denote the set of all functions $[0,1] \to \mathbb{R}$. Define a relation on $\mathscr{F}(\mathbb{R})$ by saying $f \sim g$ iff the set of all $x \in \mathbb{R}$ where $f(x) \neq g(x)$ has measure 0; in words, we say "*f equals g almost everywhere.*" Show that this defines an equivalence relation.

[Hint: Recall that an equivalence relation is reflexive, symmetric, and transitive.]

(c) Suppose $f, g\colon [0,1] \to \mathbb{R}$ are equal almost everywhere, and both are Riemann integrable. Show that

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx.$$

Problem 8. [5+5+5 points]

(a) Let $\mathscr{F} = \{f_1, \ldots, f_n\}$ be a *finite* collection of uniformly continuous functions. Prove that $\mathscr{F}$ is equicontinuous.

(b) Consider the infinite sequence of functions

$$f_n(x) = \frac{x}{x + \frac{1}{n}}, \quad x \in [0, 1],\ n \in \mathbb{N}.$$

Show that each function f_n is uniformly continuous.

(c) Show that the sequence of functions in **(b)** has no uniformly convergent subsequence. Conclude that it is not equicontinuous.

Problem 9. [10 points] Let (p_n) be a sequence in a metric space X and $p \in X$ with the following property: Every subsequence of (p_n) itself has a subsequence which converges to p. Show that $\lim_{n\to\infty} p_n = p$.

Problem 10. [This is a bonus problem – do not attempt it until you have worked all other problems] Let X be a compact metric space. Consider the metric space $(C(X), d_\infty)$ of continuous $\mathbb{R}$-valued functions on X equipped with the sup-metric

$$d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

(a) Fix some point $x_0 \in X$. Show that the subset $K_0 \subset C(X)$ of all functions f for which $f(x_0) = 0$ is closed.

(b) Let $E \subseteq X$ be a dense subset. Let $K_1 \subset C(X)$ be the subset of all functions f for which $f(e) = 0$ for all $e \in E$. Show that K_1 is closed. Can you easily describe K_1?

(c) Let $B \subset C(X)$ denote the set of continuous functions f with $\sup_{x \in X} |f(x)| < 1$. Show that B is *open* in $(C(X), d_\infty)$.

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