Practice Quiz 4

18.100B R2 Fall 2010

Closed book, no calculators.

YOUR NAME: ____

This is a 60 minute in-class exam. No notes, books, or calculators are permitted. Point values are indicated for each problem. Do all the work on these pages.

Problem 1. [15 points] Fix $n \in \mathbb{N}$ and let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{2^n} & \text{if } x = \frac{k}{2^n} \text{ for } k \text{ odd }, 0 < k < 2^n, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is Riemann integrable on $[0,1]$, and that $\displaystyle\int_0^1 f(x)\,dx = 0$.

Problem 2. [10 points] Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, nonnegative (i.e. $f(x) \geq 0$ for all $x \in [a, b]$), and $\int_a^b f(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Problem 3. [5+5+10 points]

(a) Consider the sequence of partial sums

$$f_n(x) = \sum_{k=1}^{n} e^{-kx} \cos(kx).$$

For any $a > 0$ show that f_n converges uniformly on $[a, \infty)$.

(b) Let $f(x)$ denote the limit of the sequence $f_n(x)$ in (a). Show that $f(x)$ is continuous on $(0, \infty)$.

(c) Using the function $f : (0, \infty) \to \mathbb{R}$ from (b), show that $\int_1^\infty f(x)dx$ exists. In addition, find and prove an explicit upper bound such as 2 or $\frac{e}{e-1}$. (Hint: Do not integrate anything other than e^{-kx}.)

Problem 4. [20 points: $+4$ for each correct true/false, -4 for each incorrect true/false; you can opt for 'unsure' and gain up to $+2$ for giving your thoughts.]

a) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b). Then f is Riemann integrable.

TRUE UNSURE FALSE

b) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then the function $F : [a, b] \to \mathbb{R}$ given by $F(x) = \int_a^x f(t)dt$ is continuous.

TRUE UNSURE FALSE

c) If $f : [a, b] \to \mathbb{R}$ is Riemann integrable and satisfies $f(x) < 0$ for all $x \in [a, b] \cap \mathbb{R}$, then $\int_a^b f(x)dx < 0$.

TRUE UNSURE FALSE

d) If $f_n : [a, b] \to \mathbb{R}$ is a sequence of continuous functions, and $f_n \to f$ converges uniformly, then the limit f is uniformly continuous.

TRUE UNSURE FALSE

e) If $f_n : [a, b] \to \mathbb{R}$ is a sequence of almost everywhere continuous functions, and $f_n \to f$ converges uniformly, then the limit f is almost everywhere continuous.

TRUE UNSURE FALSE