18.103 Fall 2013

Problem Set 5

AG §2.5, pp 100–102: 11, 13 (With 13, say what hypothesis of Theorem 11, page 95, failed.)

AG §2.6, pp 108–110: 2, 4, 5, 7, 8.

Further exercises

1. Consider the probability space $(I, \mathcal{M}, \mu)$ where I is the unit interval, $\mathcal{M}$ is the σ-field of Lebesgue measurable sets, and μ is Lebesgue measure. Find three subsets, A_i, $i = 1, 2, 3$, that are not independent, but which are pairwise independent.

2. **Another frequently used version of Fubini's Theorem**

a) Correct the statement of the following theorem by adding the missing hypothesis on μ and ν (c.f. 2.5/13). Then deduce it from the other versions of Fubini's theorem.

Theorem 0.1 *(Fubini, almost correct version 4) Suppose that $f(x, y)$ is a measurable function on $X \times Y$. Suppose further that*

$$\int_Y \left[\int_X |f(x,y)| d\mu(x) \right] d\nu(y) < \infty \tag{*}$$

Then f is integrable (with respect to $\mu \times \nu$ on $X \times Y$) and

$$\int_Y \left[\int_X f(x,y) d\mu(x) \right] d\nu(y) = \int_X \left[\int_Y f(x,y) d\nu(y) \right] d\mu(x) = \int_{X \times Y} f \, d(\mu \times \nu) \tag{**}$$

b) Consider the function in 2.5/12

$$f(x, y) = \frac{xy}{(x^2 + y^2)^2}$$

on the sets $X = [-1, 1]$ and $Y = [0, 1]$ with μ and ν Lebesgue measure. Show that in this case hypothesis (*) above fails, the left-hand (iterated) integral in (**) exists while the other two do not. In particular, 2.5/12a is deceptive, the result of symmetry, not Fubini's theorem.

18.103 Fourier Analysis
Fall 2013