MATH 18.152 - MIDTERM EXAM

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Midterm Exam Solutions, **Thursday, October 27**

I. **a)** Setting

(1) $$u(t,x) = v(t)w(x)$$

leads to the ODEs

(2) $$\frac{v'(t)}{v(t)} = \frac{w''(x)}{-w(x)} = \lambda \in \mathbb{R}.$$

The solutions to (2) that satisfy the boundary conditions are

(3) $$v(t) = Ae^{\lambda t},$$

(4) $$w(x) = B\sin(\sqrt{|\lambda|}x),$$

where A, B are constants,

(5) $$\lambda = m^2\pi^2,$$

and $m \geq 0$ is an integer. Thus, we have derived an infinite family of solutions

(6) $$u_m(t,x) \stackrel{\text{def}}{=} A_m e^{m^2\pi^2 t}\sin(m\pi x).$$

b) For a general $f(x)$, the solution to the PDE is a superposition

(7) $$u(t,x) = \sum_{m=1}^{\infty} A_m e^{m^2\pi^2 t}\sin(m\pi x),$$

(8) $$A_m = 2\int_{[0,1]} f(x)\sin(m\pi x)\,dx.$$

Let $n > 0$ be any integer, and consider the initial datum $f(x) = \epsilon\sin(n\pi x)$, where $\epsilon > 0$ is a small number. Then this function satisfies $\max_{x\in[0,1]}|f(x)| \leq \epsilon$, $A_m = \epsilon$ if $m = n$, and $A_m = 0$ otherwise. Thus, the corresponding solution is

(9) $$u(t,x) = \epsilon e^{n^2\pi^2 t}\sin(n\pi x).$$

At time $t = 1$, the amplitude of this solution has grown to $\epsilon e^{n^2\pi^2}$. Thus, by choosing n to be large, at $t = 1$, the solution can be arbitrarily large, even though the datum satisfies $\max_{x\in[0,1]}|f(x)| \leq \epsilon$.

In contrast, when $f = 0$, the solution remains 0 for all time. Thus, arbitrarily small changes in the data can lead to arbitrarily large changes in the solution, and the backwards heat equation is therefore *not well posed* (i.e. it is *ill-posed*). This is in complete contrast to the ordinary heat equation, which is well-posed. For the ordinary heat equation, the Fourier modes exponentially decay in time (as opposed to the exponential growth from (9)).

II. Define $v(x) = u(x) + \sqrt{R}$. Then $\Delta v = 0$, and $v(x) \geq 0$ for $x \in B_R(0)$. Thus by Harnack's inequality

$$\frac{R(R - |x|)}{(R + |x|)^2} v(0) \leq v(x) \leq \frac{R(R + |x|)}{(R - |x|)^2} v(0) \tag{10}$$

holds for $x \in B_R(0)$. Thus, for $x \in B_R(0)$, we have

$$\Big\{ \frac{R(R - |x|)}{(R + |x|)^2} - 1 \Big\} \sqrt{R} \leq u(x) \leq \Big\{ \frac{R(R + |x|)}{(R - |x|)^2} - 1 \Big\} \sqrt{R}. \tag{11}$$

For a fixed x, we let $R \to \infty$ and apply L'Hôpital's rule to conclude that

$$0 \leq u(x) \leq 0. \tag{12}$$

Thus,

$$u(x) = 0. \tag{13}$$

III. Let (t_0, x_0) be the point in $[0, 2] \times [0, 1]$ at which u achieves its max. We will show that $u(t_0, x_0) \leq 0$, which implies the desired conclusion.

If $t_0 = 0$, $x_0 = 0$, or $x_0 = 1$, then the conditions on f, g, h immediately imply that $u(t_0, x_0) \leq 0$, and we are done. So let us assume that none of these cases occur.

If $t_0 = 2$, then by the above remarks we can assume that $x_0 \in (0, 1)$. Then $\partial_t u(2, x_0) \geq 0$, since otherwise we could slightly decrease t_0 and cause the value of u to increase, which contradicts the definition of a max. Also, by standard calculus, we must have that $\partial_x u(2, x_0) = 0$, and by Taylor expanding, we can conclude that $\partial_x^2 u(2, x_0) \leq 0$ at the max. Thus, $\partial_t u(2, x_0) - \partial_x^2 u(2, x_0) \geq 0$. Using the PDE, we thus conclude that $-u(2, x_0) \geq 0$, and we are done.

For the final case, we assume that $0 < t_0 < 2$ and $x_0 \in (0, 1)$. Then by standard calculus, $\partial_t u(t_0, x_0) = 0$ at the max. Also, as above, by standard calculus $\partial_x^2 u(t_0, x_0) \leq 0$ at the max. Thus, $\partial_t u(t_0, x_0) - \partial_x^2 u(t_0, x_0) \geq 0$. Using the PDE, we thus conclude that $-u(t_0, x_0) \geq 0$, and we are again done.

IV. a)

Using the PDE and the fundamental theorem of calculus, we compute that

$$\frac{d}{dt}\mathcal{T}(t) = \int_{[0,1]} \partial_t u(t, x)\, dx = \int_{[0,1]} \partial_x^2 u(t, x)\, dx = \partial_x u(t, y)|_{y=0}^{y=1} = \partial_x u(t, 1) - \partial_x u(t, 0) = 0. \tag{14}$$

b) (2 pts.)

Our previous studies of the heat equation have suggested that solutions to the heat equation tend to rapidly settle down to constant states as $t \to \infty$. Since the thermal energy is preserved in time, if u converges to a constant C, then it must be the case that

$$C = \int_{[0,1]} C\,dy = \int_{[0,1]} \lim_{t\to\infty} u(t,y)\,dy = \lim_{t\to\infty}\int_{[0,1]} u(t,y)\,dy = \lim_{t\to\infty}\mathcal{T}(t) = \lim_{t\to\infty}\mathcal{T}(0) = \mathcal{T}(0). \tag{15}$$

c) Define $C \stackrel{\text{def}}{=} \mathcal{T}(0) = \int_{[0,1]} f(x)\,dx$. Also define $w(x) \stackrel{\text{def}}{=} u(t,x) - C$. Note that part **a)** implies that

$$\int_{[0,1]} w(t,x)\,dx = 0 \tag{16}$$

for all t.

Then we compute that

$$\begin{aligned} \partial_t w - \partial_x^2 w = 0, &\qquad (t,x) \in [0,\infty)\times[0,1], \\ w(0,x) = f(x) - C, &\qquad x \in [0,1], \\ \partial_x w(t,0) = 0, \qquad \partial_x w(t,1) = 0, &\qquad t \in [0,\infty). \end{aligned} \tag{17}$$

Define the energy

$$E^2(t) \stackrel{\text{def}}{=} \int_{[0,1]} w^2(t,x)\,dx. \tag{18}$$

Using (17) and the boundary conditions, we compute that

$$\frac{d}{dt}E^2(t) = 2\int_{[0,1]} w(t,x)\partial_t w(t,x)\,dx = 2\int_{[0,1]} w(t,x)\partial_x^2 w(t,x)\,dx = -2\int_{[0,1]} (\partial_x w(t,x))^2\,dx. \tag{19}$$

Now (16) implies that at each fixed t, there must exist a spatial point x_0 such that $w(t,x_0) = 0$; otherwise, $w(t,\cdot)$ would be strictly positive or negative in x, and therefore (16) could not hold. Using the fundamental theorem of calculus and the Cauchy-Schwarz inequality, we thus estimate that

$$\begin{aligned} |w(t,x)| = |w(t,x) - w(t,x_0)| &= \left|\int_{x_0}^x \partial_y w(t,y)\,dy\right| \\ &\leq \int_0^1 |\partial_y w(t,y)|\,dy \\ &\leq \left(\int_0^1 1^2\,dy\right)^{1/2}\left(\int_0^1 |\partial_y w(t,y)|^2\,dy\right)^{1/2} \\ &= \left(\int_0^1 |\partial_y w(t,y)|^2\,dy\right)^{1/2}. \end{aligned} \tag{20}$$

It thus follows from (20) that

$$E^2(t) \stackrel{\text{def}}{=} \int_{[0,1]} |w(t,x)|^2\,dx \leq \max_{x\in[0,1]} |w(t,x)|^2 \leq \int_0^1 |\partial_y w(t,y)|^2\,dy. \tag{21}$$

Combining (19) and (21), we have that

$$\frac{d}{dt}E^2(t) \leq -2E^2(t). \tag{22}$$

Integrating (21), we conclude that

$$E^2(t) \leq E^2(0)e^{-2t}, \tag{23}$$

and so $\lim_{t\to\infty} E^2(t) = 0$. Thus,

$$\lim_{t\to\infty} \int_{[0,1]} |u(t,x) - C|^2 \, dx = 0. \tag{24}$$

Equivalently,

$$\|u(t,\cdot) - C\|_{L^2([0,1])} \to 0 \tag{25}$$

as $t \to \infty$. That is, $u(t,\cdot)$ converges to C in the spatial $L^2([0,1])$ norm as $t \to \infty$.

V. a)

Using the PDE, integrating by parts in x, and using the spatial compact support of the solution to discard the boundary terms, we compute that

$$\begin{aligned} \frac{d}{dt}E^2 &= \int_{\mathbb{R}} \partial_t \Big\{ (\partial_t u)^2 + (\partial_x u)^2 \Big\} \, dx \\ &= 2\int_{\mathbb{R}} (\partial_t u)\partial_t^2 u + \partial_x u \partial_t \partial_x u \, dx = 2\int_{\mathbb{R}} (\partial_t u)(\partial_x^2 u - \mathfrak{F}) + (\partial_x u)\partial_t\partial_x u \, dx \\ &= -2\int_{\mathbb{R}} (\partial_t u)\mathfrak{F} \, dx + 2\int_{\mathbb{R}} (\partial_t u)\partial_x^2 u + (\partial_x u)\partial_t\partial_x u \, dx \\ &= -2\int_{\mathbb{R}} (\partial_t u)\mathfrak{F} \, dx + 2\int_{\mathbb{R}} -\partial_x(\partial_t u)\partial_x u + (\partial_x u)\partial_t\partial_x u \, dx \\ &= -2\int_{\mathbb{R}} (\partial_t u)\mathfrak{F} \, dx. \end{aligned} \tag{26}$$

b)

Using (26) and Cauchy-Schwarz, we compute that

$$\begin{aligned} \frac{d}{dt}E^2 &\leq 2\Big(\int_{\mathbb{R}} |\partial_t u|^2 \, dx\Big)^{1/2}\Big(\int_{\mathbb{R}} |\mathfrak{F}|^2 \, dx\Big)^{1/2} \\ &\leq 2E\Big(\int_{\mathbb{R}} |\mathfrak{F}|^2 \, dx\Big)^{1/2} \\ &\leq E\frac{2}{1+t^2}. \end{aligned} \tag{27}$$

But the left-hand side of (27) is equal to $2E\frac{d}{dt}E$, which leads to

$$\frac{d}{dt}E \leq \frac{1}{1+t^2}. \tag{28}$$

Integrating (28) in time, we conclude that

$$\begin{aligned} E(t) - E(0) &= \int_0^t \frac{d}{ds}E(s)\,ds \leq \int_0^t \frac{1}{1+s^2}\,ds \\ &\leq \int_0^\infty \frac{1}{1+s^2}\,ds \\ &\overset{\text{def}}{=} C < \infty, \end{aligned} \tag{29}$$

and we have reached the desired conclusion.

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