Given two sets, $\mathcal{A}$ and $\mathcal{B}$, the *Cartesian product* $\mathcal{A} \times \mathcal{B}$ is defined as the set of pairs (a, b) with $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The *size* of a pair (a, b) is defined to be the size of a plus the size of b.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be classes of objects and let $A(x)$ and $B(x)$ be their generating functions. Then the class $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ has generating function $C(x) = A(x)B(x)$.*

Proof. Let $A(x) = \sum_{j \geq 0} a_j x^j$, $B(x) = \sum_{k \geq 0} b_k x^k$, and $C(x) = \sum_{n \geq 0} c_n x^n$. To show that $C(x)$ and $A(x)B(x)$ are equal, we show that the coefficients of their series expansions are equal.

An ordered pair (a, b) of size n in $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ can be obtained by choosing an object $a \in \mathcal{A}$ of size $i \leq n$ (a_i choices) *and* an object $b \in \mathcal{B}$ of size $n - i$ (b_{n-i} choices). So

$$c_n = \sum_{i=0}^{n} a_i b_{n-i} \tag{1}$$

is the total number of ways to obtain such an ordered pair.

But the coefficients of the series expansion of $A(x)B(x)$ turn out to be the c_n from (1). Consider

$$A(x)B(x) = \left(\sum_{j \geq 0} a_j x^j \right) \times \left(\sum_{k \geq 0} b_k x^k \right).$$

In order to get the coefficient for x^n in this product, we must multiply each monomial $a_i x^i$ for $i \leq n$ from the first sum with the corresponding monomial $b_{n-i} x^{n-i}$ from the second sum. Thus we have

$$A(x)B(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) x^n = C(x).$$

We have seen that $A(x)B(x) = C(x)$ because the coefficients of their series expansions are equal.

□

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be classes of objects and let $A(x)$ and $B(x)$ be their generating functions. Then the class $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ has generating function $C(x) = A(x)B(x)$.*

Proof. To show that $C(x)$ and $A(x)B(x)$ are equal, we show that the coefficients of their series expansions are equal.

First let $C(x) = \sum_{n\geq 0} c_n x^n$. Because $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, the coefficient c_n is the total number of ways to obtain an ordered pair $(a, b) \in C$ of size n. Each pair is obtained by choosing an object $a \in \mathcal{A}$ of size $i \leq n$ (a_i choices) *and* an object $b \in \mathcal{B}$ of size $n - i$ (b_{n-i} choices). Thus the total ways to obtain such a pair is

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}. \tag{2}$$

But these c_n turn out to also be the coefficients of the series expansion of $A(x)B(x)$. If we let $A(x) = \sum_{j\geq 0} a_j x^j$ and $B(x) = \sum_{k\geq 0} b_k x^k$, then

$$A(x)B(x) = \left(\sum_{j\geq 0} a_j x^j\right) \times \left(\sum_{k\geq 0} b_k x^k\right).$$

In order to get the coefficient for x^n in this product, we must multiply each monomial $a_i x^i$ for $i \leq n$ from the first sum with the corresponding monomial $b_{n-i}x^{n-i}$ from the second sum. Thus we have

$$A(x)B(x) = \sum_{n\geq 0} \left(\sum_{i=0}^{n} a_i b_{n-i}\right) x^n.$$

Because the coefficients of this series expansion are just the c_n from (2), we have that $A(x)B(x) = \sum_{n\geq 0} c_n x^n = C(x)$. □

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