18.314 SOLUTIONS TO PRACTICE FINAL EXAM

(for Final Exam of December 15, 2014)

1. (a) (5 points) Let $F(x)=\sum_{n\geq 0} f(n)x^n$. Multiply the recurrence by x^{n+2} and sum on $n\geq 0$ to get

$$F(x)-2-4x=4x(F(x)-2)-2x^2F(x),$$

so

$$\begin{aligned} F(x) &= \frac{2-4x}{1-4x+2x^2} \\ &= \frac{1}{1-(2+\sqrt{2})x}+\frac{1}{1-(2-\sqrt{2})x}. \end{aligned}$$

Thus

$$f(n)=(2+\sqrt{2})^n+(2-\sqrt{2})^n.$$

(b) (5 points) We have $2-\sqrt{2}=0.5857\cdots$, so $0<(2-\sqrt{2})^n<1$ for all $n\geq 1$. It follows that

$$\lfloor(2+\sqrt{2})^n\rfloor=f(n)-1.$$

Now $f(1)$ is even and $f(n+2)=2(2f(n+1)-f(n))$ for $n\geq 0$, so $f(n)$ is even for $n\geq 1$. Thus $\lfloor(2+\sqrt{2})^n\rfloor$ is odd for $n\geq 1$. We can also see that $\lfloor(2+\sqrt{2})^0\rfloor=1$, which is also odd.

2. This is a situation for the exponential formula. Partition the set $[n]$ into blocks. On each block of odd size k place a cycle in $(k-1)!$ ways. In each of even size place a cycle and then color red or blue in $2(k-1)!$ ways. By the exponential formula,

$$\begin{aligned} F(x) &= \exp\left(\sum_{k\text{ odd}}(k-1)!\frac{x^k}{k!}+2\sum_{k\text{ even}}(k-1)!\frac{x^k}{k!}\right) \\ &= \exp\left(\sum_{k\geq 1}\frac{x^k}{k}+\sum_{k\geq 1}\frac{x^{2k}}{2k}\right) \\ &= \exp\left(-\log(1-x)-\frac{1}{2}\log(1-x^2)\right) \\ &= \frac{1}{(1-x)\sqrt{1-x^2}}. \end{aligned}$$

3. (a) Each tiling is a sequence of the following "primes": a 2×1 rectangle divided into two 1×1 squares, and a $2 \times k$ rectangle for $k \geq 1$. There are two primes of length one, and one prime of each length $k \geq 2$. Hence

$$\begin{aligned} F(x) &= \frac{1}{1-(2x+x^2+x^3+x^4+\cdots)} \\ &= \frac{1}{1-x-\frac{x}{1-x}} \\ &= \frac{1-x}{1-3x+x^2}. \end{aligned}$$

Note, One can easily deduce from this generating function that $f(n) = F_{2n+1}$ (a Fibonacci number), but this was not part of the problem.

(b) First consider those tilings that consist only of $2\times k$ rectangles, $k \geq 1$. The sequence of lengths of these rectangles form a composition of n. Thus the number $a(n)$ of such tilings $a(n)$ of a $2 \times n$ rectangle is 2^{n-1} $(n \geq 1)$, the number of compositions of n. Therefore

$$\begin{aligned} A(x) &:= \sum_{n\geq 1} a(n)x^n \\ &= \sum_{n\geq 1} 2^{n-1}x^n \\ &= \frac{x}{1-2x}. \end{aligned}$$

Now consider those tilings that contain no $2 \times k$ rectangle. They have a horizontal line down the middle. Above and below the line are rectangles whose lengths form a composition of n. There are $(2^{n-1})^2$ such pairs of compositions. Hence if $b(n)$ is the number of such tilings of a $2 \times n$ rectangle, then

$$\begin{aligned} B(x) &:= \sum_{n\geq 1} b(n)x^n \\ &= \sum_{n\geq 1} (2^{n-1})^2 x^n \\ &= \frac{x}{1-4x}. \end{aligned}$$

An arbitrary tiling of a $2 \times n$ rectangle consists of a sequence of tilings beginning with those counted by $a(n)$ (but which may be empty at this first step), then those counted by $b(n)$, then by $a(n)$, etc., some finite number of times. Therefore

$$\begin{aligned} G(x) &= (1+A(x))(B(x)+B(x)A(x)+B(x)A(x)B(x)+\cdots) \\ &= (1+A(x))\sum_{j\geq 0}(B(x)A(x))^j(1+B(x)) \\ &= \frac{(1+A(x))(1+B(x))}{1-A(x)B(x)}. \end{aligned}$$

Substituting $A(x) = x/(1-2x)$, $B(x) = x/(1-4x)$, and simplifying gives

$$G(x) = \frac{(1-x)(1-3x)}{1-6x+7x^2}.$$

4. If a spanning tree T does not contain the identified edge e, then there are $m+n-2$ choices, i.e., remove any of the $m+n-2$ remaining edges. If T does contain e, then we can remove any of the remaining $m-1$ edges of the m-cycle and any of the $n-1$ remaining $n-1$ edges of the n-cycle, so $(m-1)(n-1)$ choices in all. Hence

$$\kappa(G) = m+n-2+(m-1)(n-1) = mn-1.$$

A somewhat more direct argument is to remove any edge of the m-cycle and any edge of the n-cycle in mn ways. This gives a spanning tree except when we choose the identified edge e both times, so we get $mn-1$ trees in all.

5. We know (Exercise 11.12 on page 266, done in class) that G has a complete matching M. When we remove M from G we still have a regular bipartite graph (of degree $d-1 \geq 1$), so we have another matching M' disjoint from M. The union of M and M' is a disjoint union of cycles [why?].

6. The chromatic polynomial of a 4-cycle C_4 was computed in class and is easy to do in several different ways. We get

$$\chi_{C_4}(n) = n^4 - 4n^3 + 6n^2 - 3n.$$

For each of the other four vertices we have $n-2$ choices of colors. Hence

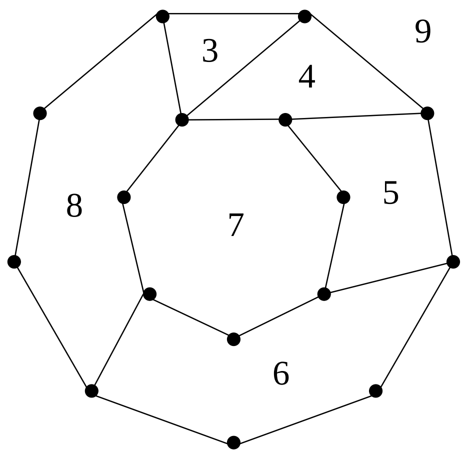
$$\chi_G(n) = (n^4 - 4n^3 + 6n^2 - 3n)(n-2)^4.$$

7. (a) If a planar embedding without isthmuses has f_i faces with i sides, then $2E = \sum i f_i$. (See equation (12.2) on page 280.) Hence

$$2E = 3 + 4 + 5 + 6 + 7 + 8 = 33,$$

contradicting that E is an integer.

(b) Now we get $2E = 3+4+5+6+7+8+9 = 42$, so $E = 21$. Since $F = 7$ we get from $V - E + F = 2$ that $V = 16$. To show that such a graph actually exists, we have to construct it. For instance, we could put the 9-sided face f on the outside and the 7-sided face completely inside f. This leads to



This is by no means the only graph meeting the conditions of the problem.

8. We claim that $n = 5$. We can easily two-color the edges of K_4 so that there is no monochromatic path of length three: color the edges of a triangle red and the remaining three edges blue. Hence $n \geq 5$. Consider now K_5 with vertices 1,2,3,4,5. The four cycle with edges 12, 23, 34, 14 must have two red and two blue edges; otherwise it already has a monochromatic path of length three. If the two red edges don't have a common vertex then one of the paths $\{12, 34, 13\}$ or $\{23, 14, 13\}$ is monochromatic. Thus we can assume that the 4-cycle has two red edges with a common vertex and two blue edges with a common vertex. Suppose that the red edges are 12,23 and the blue edges are 34,14. Then one of the paths $\{12, 23, 35\}$ and $\{34, 14, 35\}$ is monochromatic. (I'm sure there must be many other arguments.)

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