18.314: PRACTICE HOUR EXAM #1

(for hour exam of October 10, 2014)

Closed book, notes, calculators, computers, cell phones, etc. Do all four problems. Show your reasoning. There are a total of 50 points. However, the actual hour exam will have four problems and a total of 40 points.

1. (10 points) Fix an integer $n \geq 1$. Let S be the set of all n-tuples $(a_1, \ldots, a_n)$ whose entries a_i are either 1, 2, or -3. Thus $\#S = 3^n$. Find the least number $f(n)$ of elements of S we can pick so that we are *guaranteed* to have a nonempty subset T of these elements satisfying

$$\sum_{v \in T} v = (0, 0, \ldots, 0).$$

For instance, $f(2) > 3$, since no nonempty subset of the set

$$\{(1,2), (1,-3), (-3,1)\}$$

has elements summing to $(0,0)$. Note that you have to prove that your value of $f(n)$ has the stated property, and that this value is best possible, i.e., the result is false for $f(n) - 1$.

2. (10 points) Let $f(n)$ be the number of self-conjugate partitions of n, all of whose parts are even. An example of such a partition is $(4,4,2,2)$. Express $f(4n)$ in terms of $c(n)$, the total number of self-conjugate partitions of n. (Show your reasoning. A detailed proof is not necessary. Just state the basic idea.)

3. (10 points) Fix $n \geq 1$. Let $f(n)$ be the number of permutations π of $1, 2, \ldots, 2n$ with the following property: π has exactly n cycles, and the largest elements of the n cycles are the numbers $2, 4, 6, \ldots, 2n$. Find a simple formula for $f(n)$. You may write your answer either as a simple product or in terms of factorials and powers.

4. (10 points) How many partitions of the set [9] have all their blocks of size 2 or 3? You may leave your answer expressed in terms of functions discussed in class (such as binomial coefficients, factorials, Stirling numbers, etc.). You don't need to give a numerical answer.

5. (10 points) For $n \geq 1$, let $f(n)$ be the number of $n \times n$ matrices of 0's and 1's such that every row and every column has at least one 1. For instance $f(1) = 1$ and $f(2) = 7$. Use the sieve method (Principle of Inclusion-Exclusion) to give a formula for $f(n)$ as a single sum. (In my opinion this is the trickiest problem on the practice test, but I could be wrong.)

18.314 Combinatorial Analysis
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