18.314: SOLUTIONS TO PRACTICE HOUR EXAM #1

(for hour exam of October 10, 2014)

1. We can partition S into 3^{n-1} three-element blocks such that the sum of the elements in each block is $(0, 0, \ldots, 0)$. To do this define $\pi(1) = 2$, $\pi(2) = -3$, $\pi(-3) = 1$. (We are just cyclically permuting the numbers $1, 2, -3$.) Let the block containing $(a_1, a_2, \ldots, a_n)$ also contain $(\pi(a_1), \pi(a_2), \ldots, \pi(a_n))$ and $(\pi(\pi((a_1)), \pi(\pi(a_2)), \ldots, \pi(\pi(a_n)))$. For instance, when $n = 4$ one of the blocks is

$$\{(1, 2, -3, 2), (2, -3, 1, -3), (-3, 1, 2, 1)\}.$$

If we choose $2 \cdot 3^{n-1} + 1$ elements of S, then some three of them must be in the same block of the partition and therefore sum to $(0, 0, \ldots, 0)$. Thus $f(n) \leq 2 \cdot 3^{n-1} + 1$. If we choose all elements of S whose first coordinate is either 1 or 2, then the sum of any nonempty subset of the chosen elements has positive first coordinate and therefore cannot be $(0, 0, \ldots, 0)$. Since there are $2 \cdot 3^n$ vectors $(a_1, \ldots, a_n)$ with $a_1 = 1$ or 2, we see that $f(n) > 2 \cdot 3^{n-1}$. Hence $f(n) = 2 \cdot 3^{n-1} + 1$.

2. The Young diagram of a self-conjugate partition of $4n$ with even parts can be divided into n 2×2 squares. If we replace each of these 2×2 squares with a single square, then we get the Young diagram of a self-conjugate partition of n. Conversely, given the Young diagram of a self-conjugate partition of n, replace each square with a 2×2 square to get the Young diagram of a self-conjugate partition of $4n$ with even parts. Hence $f(4n) = c(n)$.

3. Insert the numbers $2, 4, 6, \ldots, 2n$, followed by $2n-1, 2n-3, \ldots, 3, 1$, in that order, into the cycle notation for π. We start with $(2\,*)(4\,*)\cdots(2n\,*)$. We always write the cycles so that $2, 4, \ldots, 2n$ are the first (leftmost) elements. Then insert $2n-1$. There is only one choice: it must be placed after $2n$. Then insert $2n-3$. There are three choices: after $2n-2, 2n-1, 2n$. Then insert $2n-5$. There are five choices: after $2n-4, 2n-3, 2n-2, 2n-1, 2n$. Continuing in this way, we see that

$$f(n) = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

Another way to write this answer is $(2n)!/2^n n!$.

4. The possible block sizes are $(3, 3, 3)$ and $(3, 2, 2, 2)$. In class it was proved that the number of partitions of $[n]$ with a_i blocks of size i is

$$\frac{n!}{1!^{a_1} 2!^{a_2} \cdots a_1!\, a_2! \cdots}.$$

Hence the number of partitions of [9] with all blocks of size 2 or 3 is equal to

$$\frac{9!}{3!^3 \cdot 3!} + \frac{9!}{2!^3 \cdot 3!^1 \cdot 1! \cdot 3!}.$$

This turns out to be equal to 1540.

5. For each subset S of $\{1, \ldots, n\}$, let $g(S)$ be the number of $n \times n$ matrices of 0's and 1's such that every row contains a 1, and if $i \in S$ then column i does not contain a 1. Each row then has $n-i$ available positions where we can place the 1's. Thus if $\#S = k$ then there are $2^{n-k}-1$ possibilities for each row. Hence $g(S) = (2^{n-k} - 1)^n$. By the sieve method,

$$\begin{aligned} f(n) &= g(\emptyset) - \sum_{\#S=1} g(S) + \sum_{\#S=2} g(S) - \cdots + (-1)^n \sum_{\#S=n} g(S) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^n. \end{aligned}$$

(The last term is 0 and can be omitted.) This problem can also be done by writing $f(n)$ as a double sum and using the binomial theorem to reduce it to a single sum. Full credit for doing it correctly this way, though the solution above is simpler.