18.314: SOLUTIONS TO PRACTICE HOUR EXAM #2

(for hour exam of November 14, 2014)

1. (a) We have

$$\begin{aligned}\sum_{n\geq 0} f(n)x^n &= \prod_{k\geq 0}\left(1+x^{2^k}+x^{2\cdot 2^k}+x^{3\cdot 2^k}\right)\\ &= \prod_{k\geq 0}\frac{1-x^{4\cdot 2^k}}{1-x^{2^k}}.\end{aligned}$$

The numerator factors cancel all the denominator factors except the first two, i.e., $1-x$ and $1-x^2$, so

$$\sum_{n\geq 0} f(n)x^n = \frac{1}{(1-x)(1-x^2)}.$$

Hence $f(n)$ is equal to the number of partitions of n with parts 1 and 2, so $S=\{1,2\}$.

(b) We want to count partitions of n into parts 1 and 2. The number of 2's in the partition can range from 0 to $\lfloor n/2\rfloor$, and the remaining parts must equal 1. Hence the number of choices is $1+\lfloor n/2\rfloor$.

2. Multiply the recurrence by x^{n+2} and sum on $n\geq 0$. Set $F(x)=\sum_{n\geq 0} a_n x^n$. We get

$$F(x)-x = 6xF(x)-8F(x),$$

so

$$\begin{aligned}F(x) &= \frac{x}{1-6x+8x^2}\\ &= \frac{x}{(1-2x)(1-4x)}\\ &= \frac{1/2}{1-4x}-\frac{1/2}{1-2x}.\end{aligned}$$

Hence $f(n)=\frac{1}{2}(4^n-2^n)$, Since $\frac{1}{2}(4^n-2^n)=\frac{1}{2}2^n(2^n-1)$, $f(n)$ is *always* a triangular number.

3. We are choosing an ordered pair (S, T) of subsets of the pencils such that $\#S$ is odd and then coloring each pencil in S either red, blue, green, or yellow, and coloring each pencil in T either white, black, or Halayà úbe. If S has k elements where k is odd, then the number of colorings of S is 4^k. If T has k elements then the number of colorings of T is 3^k. The exponential generating function for the number of colorings of S is

$$\begin{aligned} F(x) &= \sum_{k \text{ odd}} 4^k \frac{x^k}{k!} \\ &= \sinh(4x) \\ &= \frac{1}{2}(e^{4x} - e^{-4x}). \end{aligned}$$

The exponential generating function for the number of colorings of T is

$$G(x) = \sum_{k \geq 0} 3^k \frac{x^k}{k!} = e^{3x}.$$

Hence by Theorem 8.21 on page 168, we have

$$\begin{aligned} \sum_{n \geq 0} f(n) \frac{x^n}{n!} &= \frac{1}{2}(e^{4x} - e^{-4x})e^{3x} \\ &= \frac{1}{2}(e^{7x} - e^{-x}) \\ &= \frac{1}{2} \sum_{n \geq 0} (7^n - (-1)^n) \frac{x^n}{n!}, \end{aligned}$$

so $f(n) = \frac{1}{2}(7^n - (-1)^n)$.

NOTE. Halayà ubé is a kind of purple color named after a Phillipines dessert made from boiled and grated purple yams. See

http://en.wikipedia.org/wiki/List_of_colors.

4. (a) Each Hamiltonian path has $n-1$ edges. The total number of edges of K_n is $\binom{n}{2} = n(n-1)/2$. Hence the number of paths is $n/2$, which fails to be an integer when n is odd.

   (b) Let the vertices be $0, 1, \ldots, n-1$. Let one of the paths P have vertices (in their order along P)

$$0, n-1, 1, n-2, 2, n-3, \ldots, \frac{n}{2}.$$

Let the other paths be obtained from P by adding i to each coordinate for $i = 1, 2, \ldots, \frac{n}{2} - 1$, and taking the sum modulo n (i.e., if the sum exceeds $n - 1$ then subtract n from it). For instance, if $n = 8$ then the four paths are

$$\begin{array}{cccccccc} 0 & 7 & 1 & 6 & 2 & 5 & 3 & 4 \\ 1 & 0 & 2 & 7 & 3 & 6 & 4 & 5 \\ 2 & 1 & 3 & 0 & 4 & 7 & 5 & 6 \\ 3 & 2 & 4 & 1 & 5 & 0 & 6 & 7. \end{array}$$

We leave the verification that this works as an exercise.

Another way to describe the same solution (suggested by Y. Hu) is to put the vertices $0, 1, \ldots, n-1$ in clockwise order on a circle. Let P be the zigzag path whose vertices (in order) are $0, n-1, 1, n-2, 2, n-3, 3, \ldots, \frac{1}{2}n$. Rotate the circle around the center by $2j\pi/n$ radians for $0 \le j \le \frac{n}{2} - 1$. Each such rotation gives one of the paths in the partition into Hamiltonian paths.