2.050J/12.006J/18.353J Nonlinear Dynamics I: Chaos, Fall 2012

MIDTERM (At-home portion)

Problem 1: Bifurcations – a biochemical switch

A gene G, usually inactive, is activated by a biochemical substance S to produce a pigment or other gene product when the concentration S exceeds a certain threshold.

Let $g(t)$ denote the concentration of the gene product, and assume that the concentration s_0 of S is fixed. The model is

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2} \tag{1}$$

where k's are positive constants. The production of g is stimulated by s_0 at a rate k_1, and by *autocatalytic* (positive) feedback process modeled by the nonlinear term. There is also a linear degradation of g at a rate k_2.

1. Nondimensionalize the equations and bring them to the form

$$\frac{dx}{dT} = s - rx + \frac{x^2}{1+x^2}, \quad r > 0, \quad s \geq 0. \tag{2}$$

2. Show that if $s = 0$ there are two positive fixed points x_* if $r < r_c$, where r_c is to be determined.
3. Find parametric equations for the bifurcation curves in (r, s) space.
4. (MATLAB) plot quantitatively accurate plot of the stability diagram in (r, s) space
5. Classify the bifurcations that occur.
6. Assume that initially there is no gene product, i.e. $g(0) = 0$, and suppose that s is slowly increased from zero (the activating signal is turned on); What happened to $g(t)$? What happens if s then goes back to zero? Does the gene turn off again?

Problem 2: Nonlinear oscillator

Given an oscillator $\ddot{x} + b\dot{x} - kx + x^3 = 0$, b, k can be positive, negative or zero.

1. Interpret the terms physically for different values of b and k.
2. Find the bifurcation curves in (b, k) plane, state which bifurcation happens and what kind of fixed points one has to each side of the bifurcation curve.

Problem 3: Numerical study of the displaced Van der Pol oscillator

The equations for a "displaced" Van der Pol oscillator are given by

$$\dot{x} = y - a, \quad \dot{y} = -x + \delta(1 - x^2)y, \quad a > 0, \quad \delta > 0.$$

Consider a small.

1. Show that the system has two equilibrium points, one of which is a saddle. Find approximate formula for small a of this fixed point. Study this system numerically with `ode45`.

2. Submit the plots of phase plane with trajectories starting at different points (to illustrate the dynamics) for $\delta = 2$, $a = 0.1$, 0.2, 0.4, and observe that the saddle point approaches the limit cycle of the Van der Pol equation.

3. Find numerically the value of the parameter a to two decimal points when the saddle point collides with the limit cycle. What happens to the limit cycle after this collision?