18.600: Lecture 21

Joint distributions functions

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Outline

Distributions of functions of random variables

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Independent random variables

Examples

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- Suppose $P\{X \leq a\} = F_X(a)$ is known for all a. Write $Y = X^3$. What is $P\{Y \leq 27\}$?

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- If $Z = X^2$, then what is $P\{Z \leq 16\}$?

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- Given the joint distribution of X and Y, we sometimes call distribution of X (ignoring Y) and distribution of Y (ignoring X) the **marginal** distributions.
- In general, when X and Y are jointly defined discrete random variables, we write $p(x, y) = p_{X,Y}(x, y) = P\{X = x, Y = y\}$.

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- Question: if I tell you the two parameter function F, can you use it to determine the marginals F_X and F_Y?
- Answer: Yes. $F_X(a) = \lim_{b\to\infty} F(a, b)$ and $F_Y(b) = \lim_{a\to\infty} F(a, b)$.

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- From this, we can show that it works for strips, rectangles, general open sets, etc.

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- Using polar coordinates, we want $\int_0^1 (2\pi r)\frac{1}{2\pi}e^{-r^2/2}dr = -e^{-r^2/2}\Big|_0^1 = 1 - e^{-1/2} \approx .39$.

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- Can we get the marginals from that?

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- Are all of the T_i and A_i independent of each other? What are their probability distributions?

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- Negative binomial with parameters $p = 1/2$ and $n = 5$.

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- Exponential $\lambda_{\text{tiger}} = .2/\text{hour}$. So $P\{T_{\text{tiger}} > a\} = e^{-.2a}$.
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- $E[T_{\text{tiger}}] = 1/\lambda_{\text{tiger}} = 5$ hours, $\text{Var}[T_{\text{tiger}}] = 1/\lambda_{\text{tiger}}^2 = 25$ hours squared.
- Time until 5th attack by any animal?
- Γ distribution with $\alpha = 5$ and $\lambda = .6$.
- X, where Xth attack is 5th bear attack?
- Negative binomial with parameters $p = 1/2$ and $n = 5$.
- Can hiker breathe sigh of relief after 5 attack-free hours?

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- Draw the box $[0, 1] \times [0, \pi]$ on which (X, θ) is uniform. What's the area of the subset where $X \ge 1 - \sin\theta$?

18.600 Probability and Random Variables
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