18.600: Lecture 36

Call functions and Black-Scholes

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Outline

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Call function

Black-Scholes

Call function: pretty cool whether you love finance or not

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  3. $C(K)$ is an anti-anti-derivative of the density function f.

  Note that $C(0) = E[X]$ and $\lim_{K\to\infty} C(K) = 0$. C is convex with slope increasing from -1 to 0.

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- Wonder if C is good for anything....

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- **Weird fact:** If X is a real world random quantity (such as the price of gold or euros or stock shares at a future date) and we use risk neutral probability, then sometimes the call function C (or a related "put function") is what we can look up online. One then uses the quoted C values to work out F_X and f_X.

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- **Grand story goal:** Say something about the link between probability and the real world. What is the probability that price of Microsoft stock will rise by more than ten dollars over the next month? What is the probability that price of oil will drop more than ten percent next year? How can I (using internet and math) come up with a reasonable answer?

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- Implies **fundamental theorem of asset pricing**, which says discounted price $\frac{X(n)}{A(n)}$ (where A is a risk-free asset) is a martingale with respected to **risk neutral probability**.

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- Can look up $C(K)$ values for stock (say GOOG) at cboe.com, apply smoothing, take derivatives, approximate F_X and f_X.

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- For simplicity we focus on call functions in this lecture.

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  where N is normal with mean μ and variance $T\sigma^2$.
- **Surprise:** No need to guess μ. It is fixed by X_0, r, σ, T.

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- Write this as

$$e^{-rT} E[\max\{0, e^N - K\}] = e^{-rT} E[(e^N - K) 1_{N \geq \log K}]$$

$$= \frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) dx.$$

The famous formula

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- Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function Φ.
- Price of European call is $\Phi(d_1)X_0 - \Phi(d_2)Ke^{-rT}$ where $d_1 = \frac{\ln(\frac{X_0}{K}) + (r + \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln(\frac{X_0}{K}) + (r - \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$.

Perspective: implied volatility

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- Nonetheless, "implied volatility" has become a standard part of the finance lexicon. When traders want to get a rough sense of how a financial derivative is priced, they often ask for the implied volatility (a number automatically computed in many financial software packages).

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- **Where arguments for assumption break down:** Fluctuation sizes vary from day to day. Prices can have big jumps. Past volatility does not determine future volatility.
- **Fixes:** variable volatility, random interest rates, Lévy jumps....

18.600 Probability and Random Variables
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