## 18.700 Problem Set 5

Due in class Tuesday October 22; late work will not be accepted. Your work on graded problem sets should be written entirely on your own, although you may consult others before writing.

1. (6 points) Suppose we are given

$$\text{three } \textit{distinct} \text{ elements } x_1,\ x_2, \text{ and } x_3 \text{ in } F; \qquad (*)$$

and

$$\text{three } \textit{arbitrary} \text{ elements } a,\ b, \text{ and } c \text{ in } F; \qquad (**)$$

The problem is to find all polynomials

$$p(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 \qquad (***)$$

of degree less than or equal to three satisfying the conditions

$$p(x_1) = a, \qquad p(x_2) = b, \qquad p(x_3) = c. \qquad (****)$$

a) The conditions on $p$ can be written as a system of three simultaneous linear equations in four unknowns. Write the augmented matrix of this system of equations.

b) Perform elementary row operations to bring this augmented matrix to reduced row-echelon form. (This is a bit disconcerting, because some of the entries of the matrix are not "numbers" like 7, but symbols for numbers, like $x_2$. You know from algebra how to add, subtract, and multiply such symbols. What requires care is dividing: before you divide by something like $x_2$, you need to explain why it is not zero, or else worry separately about the case when it *is* zero. But you should be able to manage.

c) Write all the polynomials of degree less than or equal to three satisfying the condition $(****)$.

2. (4 points) In class I talked about the sequence of real numbers defined by $a_0 = 0$, $a_1 = 1$, and

$$a_{n+1} = a_n + 2a_{n-1} \qquad (n \geq 1).$$

I explained that this sequence has something to do with the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

a) Find all eigenvalues and eigenvectors of $A$.

b) Write $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a linear combination of eigenvectors of $A$.

c) Write a formula for $A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ not using the matrix $A$. (A good answer is a vector of formulas depending on $n$: something like $\begin{pmatrix} 2n+1 \\ n^2 \end{pmatrix}$.)

3. (3 points) Axler, page 94, exercise 19.
4. (3 points) Axler, page 95, exercise 20.
5. (3 points) Axler, page 96, exercise 21.
6. (3 points) Axler, page 96, exercise 23.

18.700 Linear Algebra
Fall 2013