HOMEWORK 10 FOR 18.725, FALL 2015
DUE THURSDAY, DECEMBER 3 BY 1PM.

(1) Suppose that Z is a closed subvariety in an affine variety X, such that $U = X \setminus Z$ is also affine. Show that for any point $x \in Z$ we have $\dim_x(Z) \geq \dim_x(X) - 1$.

[Hint: reduce to irreducible X, Z, then assuming $\dim(Z) \leq \dim(X) - 2$ construct a map from the normalization of X to U].

(2) Two smooth subvarieties Y, Z in a smooth n-dimensional variety X are said to intersect transversely at a point $x \in Y \cap Z$ is $T_x(Y) \cap T_x(Z)$ has dimension $\dim_x(Y) + \dim_x(Z) - \dim_x(X)$. They intersect transversely if they intersect transversely at every point.

(a) Show that if Y and Z as above intersect transversely then $Y \cap Z$ is a smooth subvariety and we have $I_{Y\cap Z} = I_Y + I_Z$.

(b) If Y and Z intersect transversely then $I_{Y\cup Z} = I_Y \cdot I_Z$.

(3) For a locally free coherent sheaf $\mathcal{E}$ of rank r on a variety X we write $\det(\mathcal{E})$ for the class of $\Lambda^r(\mathcal{E}) \in Pic(X)$.

(a) Show that for locally free coherent sheaves $\mathcal{E}_1$, $\mathcal{E}_2$ of ranks r_1, r_2 we have $\det(\mathcal{E}_1 \otimes \mathcal{E}_2) = \det(\mathcal{E}_1)^{r_2} \det(\mathcal{E}_2)^{r_1}$.

[Hint: choose local trivializations, then use a similar identity for determinants of matrices].

(b) Let $L \in Pic(Gr(k,n))$ be the pull-back of $\mathcal{O}(1)$ under the Plücker embedding. Find N such that $K_{Gr(k,n)} = L^N$.

(c) Find a map $\mathbb{P}^{n-k} \to Gr(k,n)$ such that the pull-back of L is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-k}}(1)$. Conclude that $L \in Pic(Gr(k,n))$ is a primitive element, i.e. $L \neq (L')^n$ for any L', $n > 1$.

[In fact $Pic(Gr(k,n)) \cong \mathbb{Z}$, so part (c) shows that L is a generator of the Picard group].

(4) Let X be a complete irreducible curve over a field k of characteristic different from 2. Assume that $f : X \to \mathbb{P}^1$ is a degree two map[1] and $i : X \to X$ is an involution, $f \circ i = f$ ($i \neq id$). Show that every section σ of K_X satisfies $i^*(\sigma) = -\sigma$.

[Hint: if $i^*(\sigma) = \sigma$ then σ vanishes on the ramification divisor, applying the Riemann-Hurwitz formula leads to a contradiction].

(5) The *dual variety* $\check{X}$ to a smooth closed subvariety $X \subset \mathbb{P}^n$ is the set of all points in $(\mathbb{P}^n)^*$ parametrizing a hyperplane tangent to X, i.e. containing the tangent space to some point in X. If X is not smooth then $\check{X}$ is defined as the closure of the set of hyperplanes tangent to a smooth point of X.

(a) Describe $\check{X}$ if X is the image of Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$.

(b) Check that if $X \subset \mathbb{P}^2$ is a smooth degree n curve over a field of characteristic zero, then $\check{X}$ is a curve of degree $n(n-1)$.

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[1] A curve X for which such an f exists is called hyperelliptic.

(c) (Optional problem) Check that if $X \subset \mathbb{P}^2$ is a smooth curve of degree 3, then $\check{X}$ has 9 simple cusp singularities, i.e. the completed local ring at each singular point is isomorphic to $k[[x,y]]/(y^2 - x^3)$.

(6) (Optional problem) This problem introduces an important construction, the deformation to normal cone. Let $X = Spec(A)$ be an affine variety and Z a closed subvariety. Let $\hat{A}$ be a $\mathbb{Z}$–graded ring, whose graded components are given by $\hat{A}_n = I_Z^n$ for $n > 0$ and $\hat{A}_n = A$ for $n < 0$, where multiplication is induced by the multiplication in A, set $\tilde{X} = Spec(\hat{A})$. Let $t \in \hat{A}$ be the element $1_A \in A_{-1} = A$.

(a) Show that the embedding $k[t] \to \hat{A}$ induces a map $\pi : \hat{X} \to \mathbb{A}^1$ such that $\pi^{-1}(\mathbb{A}^1 \setminus 0) \cong X \times (\mathbb{A}^1 \setminus 0)$ and $\pi^{-1}(0) = Spec(gr(A)_{red})$, where $gr(A) = \bigoplus_n (I_Z^n / I_Z^{n+1})$ and the subscript "red" denotes the quotient by the ideal of nilpotents.

(b) Assume that $Z = \{z\}$ is a nonisolated point. Show that $\hat{X}$ is canonically isomorphic to an open subvariety in the blow up of $X \times \mathbb{A}^1$ at $(z, 0)$.

(c) Generalize the definition of $\hat{X}$ to nonaffine varieties.

(d) The blow up of a closed subvariety Z in an affine variety X is defined as follows. If $Z = \mathbb{A}^k$ is a linear subspace in $\mathbb{A}^n = \mathbb{A}^k \times \mathbb{A}^{n-k}$, then $Bl_Z(X)$ is the product of $\mathbb{A}^k$ by the blowup of 0 in $\mathbb{A}^{n-k}$. In general we can embed X into $\mathbb{A}^n$ so that $Z = \mathbb{A}^k \cap X$, $I_Z = I_{\mathbb{A}^k} + I_X$. Then $Bl_Z(X)$ is the closure of $X \cap (\mathbb{A}^n \setminus \mathbb{A}^k)$ in $Bl_{\mathbb{A}^k}(\mathbb{A}^n)$. One can show that the closure does not depend on the auxilliary choices. Generalize part (b) to this setting.