**HOMEWORK 11 FOR 18.725, FALL 2015**
**DUE THURSDAY, DECEMBER 10 BY 1PM.**

(1) Show that a quasicoherent sheaf on a quasi-projective variety[1] $X$ is a union of its coherent subsheaves.

[Hint: reduce to the case when $X$ is projective by replacing your sheaf by its direct image under an appropriate open embedding. If $\mathcal{F}$ is a quasi-coherent sheaf on $X \subset \mathbb{P}^n$, show that every section of $\mathcal{F}|_{\mathbb{A}^n \cap X}$ extends to a map $\mathcal{O}(-d) \to \mathcal{F}$ for some $d$. Now consider the images of the direct sum of several such maps.]

(2) Recall that the arithmetic genus of a connected complete curve $X$ is the dimension of the space $H^1(\mathcal{O}_X)$.

Suppose that each component of $X$ is isomorphic to $\mathbb{P}^1$, two components intersect by at most one point and each such intersection point is a nodal singularity (i.e. its completed local ring is isomorphic to $k[[x, y]]/(xy)$).

Let $\Gamma$ be a graph whose vertices are indexed by components of $X$ and two vertices are connected by an edge when the corresponding components intersect. Show that $p_a(X) = 1 - \chi(\Gamma)$, where $p_a$ denotes the arithmetic genus and $\chi$ is the Euler characteristic.

(3) Let $X = Spec(A)$ be a normal affine irreducible surface with the only ingular point $x \in X$. Show that the following three statements are equivalent:
   (a) $Cl(X) = 0$, where $Cl$ is the divisor class group, i.e. the quotient of the group of Weil divisors by the subgroup of principal divisors.[2]
   (b) $Pic(X \setminus x) = 0$.
   (c) $A$ is UFD.

(4) Let $X$ be as in problem 3 and let $\pi : Y \to X$ be a resolution of singularities of $X$, suppose that $\pi^{-1}(X \setminus x)$ maps isomorphically to $X \setminus x$. Suppose also that the canonical line bundle $K_Y$ is trivial and that $\pi^{-1}(x)$ is a curve of the type described in problem 2, let $D_1, \dots, D_n$ be the components of $\pi^{-1}(x)$. We get a homomorphism $Pic(Y) \to \mathbb{Z}^n$, $L \mapsto (d_i)$, where the restriction of $L$ to $D_i$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d_i)$. Compute the image of (the class of) $\mathcal{O}(D_i)$ under that homomorphism.

(5) Let $G$ be a finite subgroup in $SL(2, \mathbb{C})$ and $X = \mathbb{A}^2/G$, let $x \in X$ be the image of 0. It can be shown that $X$ is normal and there exists a unique resolution $Y \to X$ satisfying the assumptions of problem 4. Moreover, the map $Pic(Y) \to \mathbb{Z}^n$ described in problem 4 is an isomorphism. Deduce that $\mathbb{C}[x, y]^G$ is a UFD iff the *Cartan matrix* constructed from the graph $\Gamma$ has determinant $\pm 1$ (in fact this determinant is always positive, so the option for it to equal $-1$ is not realized). Here Cartan matrix $C = C_\Gamma$ is given by:

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[1] This is in fact true for not necessarily quasi-projective varieties, and even more generally, see e.g. Exercise II.5.15. in Harthshorne.

[2] We have only discussed how to associate a Weil divisor to a rational function in the cases when $X$ is a curve or when $X$ is smooth. In this problem you only need to use that such a construction exists for normal irreducible varieties and that it is compatible with restriction to an open subset.

$C_{ii} = 2$, $C_{ij} = -1$ if $i \neq j$ are connected by an edge in the graph $\Gamma$ and $C_{ij} = 0$ otherwise.

[In fact, the graph $\Gamma$ is necessarily one of the simply-laced Dynkin graphs appearing in the classification of compact connected simple groups. The only such graph for which $det(C_\Gamma) = 1$ (this condition is equivalent to the corresponding simple Lie group being simply-connected) corresponds to the largest simple connected compact Lie group $E_8$. The group $G$ in this case is the binary icosahedral group, i.e. the preimage in the special unitary group $SU(2)$ of the group of symmetries of a regular icosahedron under the homomorphism $SU(2) \to PSU(2) \cong SO(3)$. The surface $X$ is isomorphic to the surface in $\mathbb{A}^3$ given by the equation $x^2 + y^3 + z^5 = 0$, as described by Felix Klein in his book "Lectures on the icosahedron and solution of the fifth degree equations" (1884); the resolution $Y$ can be obtained from $X$ by 8 blow-ups, cf. Exercise V.5.8 in Harthshorne.]

18.725 Algebraic Geometry
Fall 2015