HOMEWORK 2 FOR 18.725, FALL 2015
DUE TUESDAY, SEPTEMBER 22 BY 1PM.

(1) A Noetherian topological space X is a union of open irreducible subsets. Prove that it is irreducible if and only if it is connected.

(2) (The twisted cubic curve)

Let $X \subset \mathbb{A}^3$ be given by equations $x_2 = x_1^2$, $x_3 = x_1^3$, let C be the closure of X in $\mathbb{P}^3$.

(a) Show that $X \cong \mathbb{A}^1$, $C \cong \mathbb{P}^1$.

(b) Show that the intersection of C with a line can not contain more than two points.
[Hint: use the Vandermnonde determinant.]

(c) Let I be the ideal of polynomials vanishing on X and let J be the ideal of homogeneous polynomials vanishing on C. For a degree d polynomial P in n variables we let $\tilde{P}$ be the homogeneous degree d polynomial in $d+1$ variables given by $\tilde{P}(x_0, \dots, x_n) = x_0^d P(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$.
Show that I is generated by $P_1 = x_2 - x_1^2$, $P_2 = x_3 - x_1^3$, and that J is not generated by $\tilde{P}_1$, $\tilde{P}_2$, although J is generated by $\{\tilde{P} \mid P \in I\}$.

(3) In this problem we assume that characteristic of the base field k is different from two. For $x, y \in \mathbb{P}^{n+1}$, $x \neq y$ we let $L_{x,y}$ denote the line passing through x, y.

Let Q be an quadric in $\mathbb{P}^{n+1}$, i.e. Q is the zero set of a homogeneous square free polynomial of degree two.

(a) Show that a line L intersects Q at either one or two points, or else $L \subset Q$.

(b) Assume that Q is the zero set of a polynomial $P = \sum a_{ij} x_i x_j$, where the matrix $A = (a_{ij})$ is symmetric and nondegenerate. Pick $x \in Q$. Let U consists of points $y \in Q$ such that $y \neq x$, $L_{xy} \not\subset Q$. Show that there exists a hyperplane $H \subset \mathbb{P}^{n+1}$, such that $U = Q \setminus (Q \cap H)$.
[Hint: you can use the linear algebra fact that any nondegenerate quadratic form over an algebraically closed field k ($char(k) \neq 2$) can be sent to any other one by a linear change of variables; this allows you to reduce to the case where Q is the zero set of $x_0 x_1 + \sum_{i=2}^{n+1} x_i^2$, $x = (1 : 0 : \dots : 0)$.]

(c) Construct an isomorphism $U \cong \mathbb{A}^n$.
[Hint: One can do this by identifying the set of lines passing through x and not contained in H with $\mathbb{A}^n$.]

(d) Define a map $(Q \cap H) \setminus x \to Q_{n-2}$ where Q_{n-2} is a quadric in $\mathbb{P}^{n-1}$ so that each fiber of the map is isomorphic to $\mathbb{A}^1$.

(e) (Optional bonus problem) Let q_n be the number of points on the zero set of the polynomial $\sum_{i=0}^{n} x_{2i} x_{2i+1}$ in $\mathbb{P}^{2n+1}_{\bar{\mathbb{F}}_q}$ whose coordinates lie in $\mathbb{F}_q$. Use the previous parts of the problem to show the relation $q_n = q^{2n} + qq_{n-1} + 1$ and deduce a closed formula for q_n.