HOMEWORK 4 FOR 18.725, FALL 2015
DUE TUESDAY, OCTOBER 6 BY 1PM.

(1) Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ be a finite map.
   (a) Prove that $y \in \mathbb{A}^1$ is a ramification point iff the graph of f has an intersection with multiplicity $m > 1$ with the fiber of the second projection $\mathbb{A}^1 \times \{y\}$.
   (b) Show that if the base field k has characteristic zero then f has a ramification point unless f is an isomorphism.
   [In fact, this is true more generally for a finite morphism from an irreducible curve to $\mathbb{A}^1$].
   (c) Show that the *Artin-Schreier map* $f(x) = x^p - x$, $p = char(k)$ has no ramification points.

(2) For an algebraic variety X over a field k of characteristic p the *Frobenius twist* X' of X is defined as follows.
   $X' = X$ as a topological space. A function f on $U' \subset X'$ is regular iff $f(x) = \phi(x)^p$ where ϕ is a regular function on $U = U' \subset X$. The identity map $X \to X'$ defines a morphism $Fr : X \to X'$ called the *Frobenius morphism*.
   [Notice though that it does *not* define a morphism from X' to X.]
   (a) Check that if X is a closed subvariety in $\mathbb{A}^n$ or $\mathbb{P}^n$ whose ideal is generated by polynomials with coefficients in $\mathbb{F}_p$, then $X' \cong X$. Moreover, we have an isomorphism such that that composition $X \xrightarrow{Fr} X' \cong X$ is given by $(x_i) \mapsto (x_i^p)$.
   (b) Let X be a normal irreducible variety of dimension n. Prove that $Fr : X \to X'$ is finite, find its degree and prove that every point is its ramification point.
   [Hint: reduce to the case of $X = \mathbb{A}^n$].
   (c) Describe the intersection points of the graph of Frobenius $Fr : \mathbb{A}^1 \to \mathbb{A}^1$ with the diagonal and check that each one has multiplicity one.

(3) Let X be the line with a double point at zero, thus we have a map $X \to \mathbb{A}^1$ which is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$ and the preimage of 0 consists of two points.
   (a) Let $Y = \mathbb{A}^2 \setminus \{0\}$. Show that the map $m : Y \to \mathbb{A}^1$, $m(x, y) = xy$ can be lifted to an onto map $Y \to X$; moreover, there are two distinct such liftings.
   (b) Describe the closure of the diagonal in X^2 and X^3. More precisely, define a map from that closure to $\mathbb{A}^1$, which is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$ and count the number of points in the preimage of zero.

(4) $X \subset \mathbb{A}^{n+1}$ is the zero set of a polynomial P which is irreducible and has the form $P = P_d + P_{d+1}$ where P_d, P_{d+1} are nonzero homogeneous polynomials of degrees d, $d + 1$ respectively. Prove that X is birationally equivalent to $\mathbb{A}^n$.

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