HOMEWORK 5 FOR 18.725, FALL 2015
DUE TUESDAY, OCTOBER 20 BY 1PM.

(1) Prove that a quasi-affine complete variety is finite.

(2) Show that a quasiaffine variety Y is isomorphic to a closed subvariety in the complement to a linear subspace $\mathbb{A}^m$ in the affine space $\mathbb{A}^n$.

(3) Check that the intersection of the two quadrics in $\mathbb{P}^3$ given respectively by the equations $xy = zw$ and $x^2 = yw$ is the union of the twisted cubic and a line.

(4) (Jouanolou trick for projective varieties)
   (a) Let Y be the space of matrices $E \in Mat_n(k)$ such that $E^2 = E$ and $rank(E) = 1$.
   Show that the Y is a closed subvariety in $\mathbb{A}^{n^2} = Mat_n(k)$.
   [Hint: instead of the open condition $E \neq 0$ impose the closed condition $det(1 - E) = 0$].
   (b) Let X be a projective variety. Prove that there exists an affine variety Y and an onto map $Y \to X$ whose fibers are isomorphic to an affine space $\mathbb{A}^N$.
   [Hint: reduce it to $X = \mathbb{P}^{n-1}$, then show that Y from part (a) will do].

(5) $X_{n,m}$ be the curve $y^n = x^m$, $(n, m) = 1$.
   (a) Show that the blow up $Bl_0(X)$ is isomorphic to $X_{n',m'}$ for some n', m' and compute n', m'.
   (b) Show that applying a sequence of blow-ups to $X_{n,m}$ one can end up with a curve isomorphic to $\mathbb{A}^1$.

(6) Let $X = Spec(A)$ be an affine variety with an action of a finite group G. Let $Y = Spec(A^G)$, where A^G denotes the subring of G-invariants. (A Theorem by E. Noether asserts that A^G is finitely generated, thus the algebraic variety Y is well defined). We have a morphism $X \to Y$.
   (a) Show that for a variety T we have a canonical bijection $Hom(Y, T) \cong Hom(X, T)^G$ sending a map $Y \to T$ to the composition $X \to Y \to T$. This is expressed by saying that Y is a *categorical quotient* of X by G.
   (b) Let $G = \mathbb{Z}/3\mathbb{Z}$ act on $\mathbb{A}^2$ so that the generator acts by $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$, where ζ is a primitive cubic root of 1 (we assume that $char(k) \neq 3$). Show that $X = \mathbb{A}^2/\mathbb{Z}_3$ is isomorphic to a degree 3 hypersurface X in $\mathbb{A}^3$ of the type considered in problem 4, pset 4.
   (c) Let $x \in X$ be the image of 0 in X and $\hat{X} = Bl_x(X)$ be the blow-up of X at x. Describe the preimage of x in $\hat{X}$.
   (d) (Optional bonus problem) Show that for some point $y \in \hat{X}$ the variety $Bl_y(\hat{X})$ can be presented as a union of affine open subsets isomorphic to $\mathbb{A}^2$.

(7) (Optional bonus problem) Consider the rational map $\mathbb{P}^2 \to \mathbb{P}^4$ given by
   $\phi(x_0 : x_1 : x_2) = (x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2)$.

Prove that ϕ is birational map from $\mathbb{P}^2$ to a (closed) surface $X \subset \mathbb{P}^4$, that X is identified with the blow-up of $(1:0:0)$; in particular, the inverse map $X \to \mathbb{P}^2$ is regular.

(8) (Optional bonus problem: Jouanolou trick in general)

(a) Prove that a quasiprojective variety X is isomorphic to a closed subvariety in the complement to a linear subspace $\mathbb{P}^m$ in the projective space $\mathbb{P}^n$.
[Hint: X is closed in $U = \mathbb{P}^n \setminus Z$ where Z is the zero set of homogeneous polynomials $f_1, \ldots, f_m$. Without loss of generality we can assume that $f_1, \ldots, f_m$ have the same degree. All monomials of the given degree define a map $\mathbb{P}^n \to \mathbb{P}^N$, and Z is the preimage of the linear subspace under that map. Now U is realized as a closed subvariety in $\mathbb{P}^n \times (\mathbb{P}^N \setminus \mathbb{P}^{N'})$; now apply Veronese embedding and check that $\mathbb{P}^n \times \mathbb{P}^{N'}$ is the preimage of a linear subspace].

(b) Let M be the space of matrices $E \in Mat_{n+1}(k)$ such that $E^2 = E$, $rank(E) = 1$ and $E|_{k^{m+1}} = 0$. Show that the M is a closed subvariety in $\mathbb{A}^{(n+1)^2} = Mat_{n+1}(k)$.

(c) Define a map $M \to (\mathbb{P}^n \setminus \mathbb{P}^m)$ and show that its fibers are isomorphic to an affine space $\mathbb{A}^{n-m-1}$.

(d) Conclude that for any quasiprojective variety X there exists an affine variety Y with an onto morphism $Y \to X$ whose fibers are isomorphsic to $\mathbb{A}^N$.