# HOMEWORK 6 FOR 18.725, FALL 2015
# DUE TUESDAY, OCTOBER 27 BY 1PM.

(1) Let $Y$ be the cone over the twisted cubic blown-up at 0. Let $X$ be the blow-up of $\mathbb{A}^2$ at zero. Define an action of $G = \mathbb{Z}/3\mathbb{Z}$ on $X$, so that $X//G \cong Y$. (Here we assume that $char(k) \neq 3$, where $k$ is the ground field).

(2) Let $X \subset \mathbb{P}^1 \times \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at 0, let $\pi_2 : X \to \mathbb{A}^2$ and $\pi_1 : X \to \mathbb{P}^1$ be the projections. For each $n \in \mathbb{Z}$ describe the fiber of the sheaf $\pi_{2*}\pi_1^*\mathcal{O}_{\mathbb{P}^1}(n)$ at zero (in particular, compute its dimension).

(3) Let $G$ be a finite group acting on a quasiprojective variety $X$ (you may assume $X$ is affine). An *equivariant sheaf* on $X$ is a sheaf $\mathcal{F}$ together with isomorphisms $I_g : \mathcal{F} \cong g^*(\mathcal{F})$ fixed for each $g \in G$, so that for each $g, h \in G$ we have $I_{gh} = h^*(I_g) \circ I_h$.

   Let $QCoh^G(X)$ denote the category of $G$-equivariant quasicoherent sheaves on $X$. Let $Y = X//G$ be the (categorical) quotient and $p : X \to Y$ be the canonical map.

   The functor $p^* : QCoh(Y) \to QCoh(X)$ lifts to a functor $p_G^* : QCoh(Y) \to QCoh^G(X)$. Show that $p_G^*$ is fully faithful, describe its right adjoint and give an example showing that $p_G^*$ is not essentially surjective. (Here we assume that $|G|$ is invertible in $k$).

(4) Let $X$ be an irreducible variety.
   (a) Show that the sheaf $\mathcal{K} = \varinjlim_U j_*(\mathcal{O}_U)$, where the limit is taken over the poset of nonempty open subsets, is a constant sheaf, describe its stalks.
   (b) We have a natural map $\mathcal{O}_X \to \mathcal{K}$. If $X$ is a curve show that $\mathcal{K}/\mathcal{O}$ is an (infinite) direct sum of sheaves supported at points of $X$.
   (c) If $X = \mathbb{P}^1$, show that the sequence
   $$0 \to \Gamma(\mathcal{O}) \to \Gamma(\mathcal{K}) \to \Gamma(\mathcal{K}/\mathcal{O}) \to 0$$
   is exact.
   (d) (Optional bonus problem) Let $X \subset \mathbb{P}^2$ be given by the equation $x^3 + z^2x = y^2z$. Check that the map $\Gamma(\mathcal{K}) \to \Gamma(\mathcal{K}/\mathcal{O})$ has a one dimensional cockernel.

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