HOMEWORK 7 FOR 18.725, FALL 2015
DUE THURSDAY, NOVEMBER 5 BY 1PM.

(1) The Cremona (or quadratic) transformation[1] is a rational morphism $\mathbb{P}^2 \to \mathbb{P}^2$ given by $\phi : (t_0 : t_1 : t_2) \mapsto (t_0t_1 : t_0t_2 : t_1t_2)$.
  (a) Show that ϕ is birational and find its inverse.
  (b) Find maximal open subsets $U, V \subset \mathbb{P}^2$, such that ϕ induces an isomorphism $U \to V$.

(2) (Optional problem) The *Fermat cubic* surface $X \subset \mathbb{P}^3$ is given by the equation $x^3 + y^3 + z^3 + w^3 = 0$.
  (a) Check that the rational map[2] sending $(x : y : z : w)$ with $w + y \neq 0$ or $x + z \neq 0$ to

$$(yz - wx : wy - wx + xz + w^2 - wz + z^2 : y^2 - xy + wy + x^2 - wx + xz),$$

  is a birational isomorphism between X and $\mathbb{P}^2$.
  (b) Deduce that the Cremona group (defined in footnote 1) contains a subgroup $S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$.

(3) Let $\mathcal{F}_n \in Coh(\mathbb{A}^1)$ be the cokernel of the map $\mathcal{O} \stackrel{t^n}{\to} \mathcal{O}$ (where t is the coordinate on $\mathbb{A}^1$). We have a surjective map $\mathcal{F}_{n+1} \to \mathcal{F}_n$. Show that the inverse limit $\varprojlim_{Sh} \mathcal{F}_n$ in the category of sheaves of $\mathcal{O}$-modules is not isomorphic to the inverse limit $\varprojlim_{QCoh} \mathcal{F}_n$ in the category of quasicoherent sheaves (though both limits exist). Moreover, check that $\varprojlim_{Sh} \mathcal{F}_n$ is a non-quasicoherent sheaf supported at zero, while $\varprojlim_{QCoh}(\mathcal{F}_n)$ has full support.

(4) (a) Show that $QCoh(\mathbb{P}^1)$ contains no projective object $\mathcal{P}$ with a nonzero map $\mathcal{P} \to \mathcal{O}_{x_0}$, $x_0 \in \mathbb{P}^1$, where $\mathcal{O}_{x_0}$ is the sky-scraper at a point x_0, i.e. the direct image of $\mathcal{O}$ under the embedding $x_0 \to \mathbb{P}^1$.
  [Hint: You can use a result to be proved in class that a quasicoherent sheaf is a union of its coherent subsheaves. If $\mathcal{P}$ is a projective object apply $Hom(\mathcal{P},\ \ )$ to the surjection $\mathcal{O}(-n) \to \mathcal{O}_{x_0}$. Get a map $\mathcal{P} \to \mathcal{O}(-n)$, which must be nonzero on a coherent subsheaf $\mathcal{F} \subset \mathcal{P}$ surjecting to $\mathcal{O}_{x_0}$. Now take $n \gg 0$ and get a contradiction].
  (b) (Optional problem) Show that $j_*(\mathcal{O})/\mathcal{O}$ is an injective object in $QCoh(\mathbb{P}^1)$, where $j : \mathbb{A}^1 \to \mathbb{P}^1$ is the embedding.

(5) For a subvariety $X \subset \mathbb{P}^n$ not contained in a linear subspace, the k-th secant variety $S_k(X)$ is the closure of the union of all k-planes in $\mathbb{P}^n$ spanned by $k + 1$ points of X.
  For $n = 2k$ let $C \subset \mathbb{P}^n$ be the image of the n-th Veronese embedding of $\mathbb{P}^1$. Show that $S_{k-1}(C)$ is a hypersurface of degree $k + 1$ in $\mathbb{P}^n$.

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[1] A theorem by Noether and Castelnuovo asserts that the *Cremona group* of birational automorphisms of $\mathbb{P}^2$ is generated by ϕ and the subgroup $PGL_3(k)$ of linear automorphisms of $\mathbb{P}^2$.

[2] Formula borrowed from Noam Elkies' homepage.

[Hint: For a two-dimensional vector space V one needs to find an equation singling out elements in $Sym^n(V)$ which are sums of at most k monomials $\sum_{i=1}^k v_i^n$. An element $\sigma \in Sym^n(V)$ determines a map $Sym^k(V^*) \to Sym^k(V)$. Check that if σ is of the form $\sum_{i=1}^k v_i^n$ then the map has zero determinant, while for some element in $Sym^n(V)$ the determinant is nonzero. This shows $S_{k-1}(C)$ is contained in the zero set of a degree $k+1$ polynomial. One can also check directly that $S_{k-1}(C)$ is an irreducible hypersurface, the above shows its degree is at most $k+1$. To show it can't be smaller than $k+1$ one can write down a line intersecting it in $k+1$ points.]

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