HOMEWORK 9 FOR 18.725, FALL 2015
DUE THURSDAY, NOVEMBER 19 BY 1PM.

(1) Show that an open embedding of one dimensional varieties is an affine morphism. Conclude that every open subset in an affine curve is affine.

(2) Suppose that the zero set of polynomials $P_1, \cdots, P_m \in k[x_1, \ldots, x_N]$ is a smooth n-dimensional irreducible subvariety $X \subset \mathbb{A}^N$. Assume also that for every $x \in X$ the matrix $\left(\frac{\partial P_i}{\partial x_j}\right)$ has rank $N-n$. Show that polynomials P_i generate the ideal of X.

(3) Show that if a nonlinear hypersurface $X \subset \mathbb{P}^n$ contains a linear subspace of dimension $r \geq \frac{n}{2}$ then X is singular.

(4) A resolution of singularities for a singular irreducible variety Y is a map $\pi : X \to Y$ such that π is projective,[1] birational, while X is smooth.

Let Y be the quadratic cone $x^2 + y^2 + z^2 + t^2 = 0$ in $\mathbb{A}^4$.

(a) Show that $X = \hat{Y}$, the blow-up of zero in Y, is a resolution of singularities of Y. Check also that the preimage of 0 in X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

(b) (Optional problem) Find two nonisomorphic[2] resolutions of singularities X_1, X_2 of Y, such that the preimage of zero is isomorphic to $\mathbb{P}^1$.

(5) (a) Determine the singular points of the Steiner (or Roman) surface $S \subset \mathbb{P}^3$ given by $x_1^2x_2^2 + x_2^2x_0^2 + x_0^2x_1^2 - x_0x_1x_2x_3 = 0$. Describe its tangent cone at $(0:0:0:1)$.

(b) (Optional problem) Show that S is the image of $\mathbb{P}^2 \subset \mathbb{P}^5$ under a linear rational map $\mathbb{P}^5 \dashrightarrow \mathbb{P}^3$; here $\mathbb{P}^2 \subset \mathbb{P}^5$ is the image of the second Veronese embedding.

(6) (Optional problem)

(a) Let X be a smooth complete surface. Show that $Pic(X)$ carries a symmetric bilinear form, such that for irreducible divisors D_1, D_2 we have

$$\langle [D_1], [D_2] \rangle = \deg(\mathcal{O}(D_1)|_{D_2}) = \dim \Gamma(\mathcal{O}_{D_1} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_2}),$$

where the second equality applies only if $D_1 \neq D_2$.

The quotient of $Pic(X)$ by the kernel of this form is called the *Neron-Severi* group of X.

(b) Show that the Picard group of Fermat quartic $x_0^4 + x_1^4 + x_2^4 + x_3^4$ in $\mathbb{P}^3$ contains[3] a free abelian group of rank 20.

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[1]This means that π can be decomposed as a composition of a closed embedding to $Y \times \mathbb{P}^N$ for some N and projection to Y.

[2]The two varieties X_1 and X_2 may be isomorphic, they are required to be nonisomorphic as resolutions, i.e. there is no isomorphism $X_1 \cong X_2$ compatible with the map to Y.

[3]In fact, a smooth quartic in $\mathbb{P}^3$ is an example of a $K3$ surface, i.e. its canonical line bundle is trivial and (for $k = \mathbb{C}$) the corresponding complex manifold is simply-connected. The Picard

[Hint: it is easy to write down many explicit divisors on X, use (a) to find 20 of those such that the bilinear form in (a) is nondegenerate on their span].

group of a $K3$ surface is known to be a free abelian group whose rank r satisfies $1 \leq r \leq 20$. Thus Picard group of the Fermat quartic is isomorphic to $\mathbb{Z}^{20}$

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