18.785 Number Theory **Fall 2019**

Problem Set #9

Description

These problems are related to the material covered in Lectures 16–18. Collaboration is permitted/encouraged, but you must identify your collaborators, and any references you consulted. If there are none, write "**Sources consulted: none**" at the top of your problem set. The first person to spot each non-trivial typo/error in any of the problem sets or lecture notes will receive 1-5 points of extra credit.

Instructions: First do the warm up problems, then pick two of problems 1–5 to solve and write up your answers in latex, then complete the survey problem 6.

Problem 0.

These are warm up problems that do not need to be turned in.

(a) Show $\pi(x) := \sum_{p\le x} 1 = \int_{2^-}^{\infty}(1/\log t)d\vartheta(t)$ and $\vartheta(x) := \sum_{p\le x}\log p = \int_{2^-}^{\infty}\log t\, d\pi(t)$, and use these identities to prove

$$\vartheta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt, \qquad \pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t\log^2 t}dt,$$

which provides an alternative proof that $\pi(x) \sim x/\log x$ if and only if $\vartheta(x) \sim x$.

(b) Let χ be a primitive Dirichlet character of conductor $m > 1$. Verify the identity

$$\sum_{n\ge 1}\chi(n)x^n = \frac{1}{1-x^m}\sum_{n=1}^{m-1}\chi(n)x^n$$

and use this to prove that $\Gamma(s)L(s,\chi)$ extends to a holomorphic function on $\mathbb{C}$. Conclude that $L(s,\chi)$ has an analytic continuation to $\mathbb{C}$.

Problem 1. Mertens' Theorems (49 points)

In his 1874 paper Mertens' proved three asymptotic bounds on sums over primes; he necessarily did not rely on the Prime Number Theorem, which wasn't proved until 1896.

Define the constants

$$\alpha := -\sum_{n\ge 2}\frac{\mu(n)}{n}\log\zeta(n) \approx 0.315718, \qquad \gamma := \lim_{x\to\infty}\left(\sum_{1\le n\le x}\frac{1}{n} - \log x\right) \approx 0.577216,$$

where $\mu(n)$ is the Möbius function from Problem Set 8:

$$\mu(n) := \begin{cases} (-1)^{\#\{p|n\}} & \text{if } n\ge 1 \text{ is square free;}\\ 0 & \text{otherwise.}\end{cases}$$

Let $\Lambda(n)$ denote the *von Mangoldt function*:

$$\Lambda(n) := \begin{cases} \log p & \text{if } n > 1 \text{ is a power of a prime } p;\\ 0 & \text{otherwise.}\end{cases}$$

Theorem (Mertens). *As $x \to \infty$ we have the following asymptotic bounds:*

(1) $\sum_{p \le x} \frac{\log p}{p} = \log x + O(1)$;

(2) $\sum_{p \le x} \frac{1}{p} = \log\log x + \gamma - \alpha + O\left(\frac{1}{\log x}\right)$;

(3) $\sum_{p \le x} \log\left(1 - \frac{1}{p}\right) = -\log\log x - \gamma + O\left(\frac{1}{\log x}\right)$.

Remark. Mertens showed that the $O(1)$ term in (1) has absolute value bounded by 2, but we will not need this. One often sees (3) written as $\prod_{p \le x}(1 - \frac{1}{p}) = \frac{e^{-\gamma + o(1)}}{\log x}$ but our version is a slightly sharper statement that reflects what Mertens actually proved.

(a) Show that $\log(n) = \sum_{d|n} \Lambda(d)$ and derive the bounds

$$\sum_{n \le x} \log n = \sum_{d \le x} \Lambda(d) \lfloor \tfrac{x}{d} \rfloor \qquad \text{and} \qquad \sum_{d \le x} \frac{\Lambda(d)}{d} = \log x + O(1).$$

Use these bounds and Stirling's formula to prove (1).

(b) Let $A(x)$ denote the sum in (1). Prove that

$$\sum_{p \le x} \frac{1}{p} = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t(\log t)^2} dt = \log\log x + c + O\left(\tfrac{1}{\log x}\right),$$

for some constant c.

(c) Prove that for $\mathrm{Re}(s) > 1$ we have

$$\frac{1}{s} \log \zeta(s) = \int_2^\infty \frac{\pi(t)\, dt}{t(t^s - 1)},$$

and for $t > 1$ we have

$$\frac{1}{t^2(t-1)} = -\sum_{n \ge 2} \frac{\mu(n)}{t(t^n - 1)}.$$

(d) Prove that

$$\sum_{n \ge 2} \sum_p \frac{1}{np^n} = \int_2^\infty \frac{\pi(t)\, dt}{t^2(t-1)} = \alpha$$

and deduce that (2) and (3) are equivalent.

Remark. Parts (b) and (d) imply that (3) holds if we replace γ with $c' = c + \alpha$. Problem 2 gives a proof that in fact $c' = \gamma$, so both (2) and (3) hold.

(e) Let $P(x) := \sum_{p \le x} \frac{1}{p} = \log\log x + c + \epsilon(x)$ with $\epsilon(x) = O\left(\frac{1}{\log x}\right)$ as in (b). Show that

$$\pi(x) = \int_{2^-}^x t\, dP(t) = O\left(\frac{x}{\log x}\right),$$

and that with the error bound $\epsilon(x) = o\left(\frac{1}{\log x}\right)$ one obtains $\pi(x) \sim \frac{x}{\log x}$. Thus a slightly stronger version of Mertens' 2nd theorem implies the prime number theorem.

Problem 2. Mellin transforms of Dirichlet series (49 points)

Associated to any arithmetic function $f\colon \mathbb{Z}_{n\geq 1} \to \mathbb{C}$ is a Dirichlet series

$$D_f(s) := \sum_{n\geq 1} f(n)n^{-s},$$

which we may view a function of the complex variable s on any region $\mathrm{Re}(s) > \sigma \geq 0$ in which the series converges; conversely, the coefficients of a Dirichlet series define an arithmetic function.

We also have the *summatory function* $S_f\colon \mathbb{R} \to \mathbb{C}$ associated to f, defined by

$$S_f(x) := \sum_{1\leq n\leq x} f(n),$$

and the *logarithmic summatory function* $L_f\colon \mathbb{R} \to \mathbb{C}$ defined by

$$L_f(x) := \sum_{1\leq n\leq x} \frac{f(n)}{n}.$$

(a) Show that $D_f(s)$ is related to $S_f(x)$ and $L_f(x)$ via the formulas

$$D_f(s) = s\int_1^\infty S_f(t)t^{-s-1}\,dt \qquad (\mathrm{Re}(s) > \max(0,\sigma)),$$

$$D_f(s) = (s-1)\int_1^\infty L_f(t)t^{-s}\,dt \qquad (\mathrm{Re}(s) > \max(1,\sigma)).$$

(b) By applying (a) to $f = 1$, show that

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{t\}t^{-s-1}\,dt \qquad (\mathrm{Re}(s) > 0),$$

where $\{t\} := t - \lfloor t\rfloor$. Use this to show that as $s \to 1$ we have

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|).$$

(c) Let

$$P(x) := -\sum_{p\leq x} \log(1 - \tfrac{1}{p})$$

be the negation of the sum in Mertens' 3rd theorem (see Problem 1), and let $\kappa(n)$ be the arithmetic function defined by $\kappa(n) = 1/k$ when $n = p^k$ is a prime power ($k \geq 1$) and $\kappa(n) = 0$ otherwise (as in Problem 4.e on Problem set 8). Show that

$$P(x) = L_\kappa(x) + O\left(\tfrac{1}{\log x}\right).$$

(d) Show that $\log\zeta(s) = D_\kappa(s)$ and use (b) to prove that

$$D_\kappa(s) = \log\frac{1}{s-1} + O(s-1)$$

as $s \to 1^+$ (along the real line).

From parts (b) and (d) of Problem 1 we know that

$$P(x) = \log\log x + C + O\left(\tfrac{1}{\log x}\right) \tag{1}$$

for some constant C which, according to Mertens' 3rd theorem, is equal to Euler's constant γ. You are now in a position to prove this.

(e) From (c) and (**??**) we know that $L_\kappa = \log\log x + C + O\left(\frac{1}{\log x}\right)$. By plugging this into to the formula relating D_κ and L_κ from (a), show that we have

$$D_\kappa(s) = \log\frac{1}{s-1} + C + \int_0^\infty (\log t)e^{-t}dt + O\left((s-1)\log\tfrac{1}{s-1}\right)$$

as $s \to 1^+$.

(f) By combining (d) and (e) and letting $s \to 1^+$ show that

$$C = -\int_0^\infty (\log t)e^{-t}dt.$$

Then show that the integral is equal to $\Gamma'(1)$, and prove that $\Gamma'(1) = -\gamma$ (you can do this either by using (b) and the functional equation for $\zeta(s)$, or by evaluating the *digamma function* $\Psi(s) := \Gamma'(s)/\Gamma(s)$ at 1).

Problem 3. Dirichlet density (49 points)

Let K be a global field and let $\mathcal{P}$ be the set of nonzero prime ideals of $\mathcal{O}_K$. The *natural density* of a set $S \subseteq \mathcal{P}$ is defined by

$$\delta(S) := \lim_{x\to\infty} \frac{\#\{\mathfrak{p} \in S : \mathrm{N}(\mathfrak{p}) \le x\}}{\#\{\mathfrak{p} \in \mathcal{P} : \mathrm{N}(\mathfrak{p}) \le x\}}$$

(whenever this limit exists), and its *Dirichlet density* is defined by

$$d(S) := \lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in S} \mathrm{N}(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in \mathcal{P}} \mathrm{N}(\mathfrak{p})^{-s}}$$

(whenever this limit exists). Here $\mathrm{N}(\mathfrak{p}) := [\mathcal{O}_K : \mathfrak{p}]$ is the absolute norm.

(a) Show that the denominator in $d(S)$ is finite for real $s > 1$ and that

$$\sum_{\mathfrak{p}\in\mathcal{P}} \mathrm{N}(\mathfrak{p})^{-s} \sim \log\left(\frac{1}{s-1}\right)$$

as $s \to 1^+$.

(b) Let S and T be subsets of $\mathcal{P}$ with Dirichlet densities. Show that $S \subseteq T$ implies $d(S) \le d(T)$, and that $d(S) = 0$ when S is finite. Conclude that if S and T differ by a finite set (that is, the sets $S - T$ and $T - S$ are both finite), then $d(S) = d(T)$.

(c) Suppose $S, T \subset \mathcal{P}$ have finite intersection. Show that if any two of the set S, T, and $S \cup T$ have a Dirichlet density then so does the third and $d(S \cup T) = d(S) + d(T)$.

(d) Suppose K is a number field and define $\mathcal{P}_1 := \{\mathfrak{p} \in \mathcal{P} : \mathrm{N}(\mathfrak{p}) \text{ is prime}\}$. Show that $d(\mathcal{P}_1) = 1$ and in particular, that there are infinitely many degree one primes of K.

(e) With K and $\mathcal{P}_1$ as in (d) show for any $S \subseteq \mathcal{P}$, if S has a Dirichlet density then $d(S) = d(S \cap \mathcal{P}_1)$ and otherwise $S \cap \mathcal{P}_1$ does not have a Dirichlet density. Compute the density of the set of primes of $\mathbb{Q}(i)$ that lie above a prime $p \equiv 3 \bmod 4$.

(f) Show that if $S \subseteq \mathcal{P}$ has a natural density then it has Dirichlet density $d(S) = \delta(S)$.

(g) Show that for $K = \mathbb{F}_q(t)$ the set of primes (f) where f is an irreducible polynomial of even degree has Dirichlet density $1/2$ but no natural density.

(h) Show that for $K = \mathbb{Q}$ the set S_1 of primes whose leading decimal digit is equal to 1 has no natural density.

(i) Let A be the set of positive integers with leading decimal digit equal to 1. Show that

$$\lim_{s\to 1^+} \frac{\sum_{n\in A} n^{-s}}{\frac{1}{s-1}} = \lim_{s\to 1^+} \frac{\sum_{n\in A} n^{-s}}{\sum_{n\geq 1} n^{-s}} = \log_{10}(2).$$

(j) Adapt your argument in (i) to show that $d(S_1) = \log_{10}(2)$.

Problem 4. PNT for arithmetic progressions (49 points)

For each integer $m > 1$ and integer a coprime to m we define the prime counting function

$$\pi(x; m, a) := \sum_{\substack{p\leq x \\ p\equiv a \bmod m}} 1.$$

In this problem you will adapt the proof of the PNT in [?] (which is essentially the same as given in class except for the argument to show that $\zeta(s)$ has no zeros on $\mathrm{Re}(s) = 1$) to prove the PNT for arithmetic progressions, which states that

$$\pi(x; m, a) \sim \frac{\pi(x)}{\phi(m)} \sim \frac{1}{\phi(m)} \frac{x}{\log x},$$

where $\phi(m) := \#(\mathbb{Z}/m\mathbb{Z})^\times$ is the Euler function. We first set some notation.

Let χ denote a primitive Dirichlet character of conductor dividing m and define

$$L(s,\chi) := \sum_{n\geq 1} \chi(n) n^{-s}, \qquad \theta_{m,a}(x) := \phi(m) \sum_{\substack{p\leq x \\ p\equiv a \bmod m}} \log p,$$

$$\phi(s,\chi) := \sum_p \chi(p) p^{-s} \log p, \quad \Phi_m(s) := \sum_\chi \phi(s,\chi), \quad \Phi_{m,a}(s) := \sum_\chi \overline{\chi(a)} \phi(s,\chi).$$

Finally, let $K = \mathbb{Q}(\zeta_m)$ be the mth cyclotomic field with Dedekind zeta function $\zeta_K(s)$.

(a) Show that $\theta_{m,a}(x) = O(x)$.

(b) Show that for each χ we have

$$-\frac{L'(s,\chi)}{L(s,\chi)} = \phi(s,\chi) + h(s,\chi),$$

for some $h(s,\chi)$ holomorphic on $\mathrm{Re}(s) > 1/2$, and conclude that

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \Phi_m(s) + h(s),$$

for some $h(s)$ holomorphic on $\mathrm{Re}(s) > 1/2$.

(c) Show that $\zeta_K(s)$ is real-valued on real values of s and proceed as in step (IV) of [**?**] to show that $\zeta_K(s)$, and therefore each $L(s,\chi)$, has no zeros on $\mathrm{Re}(s) = 1$.

(d) Show that $\Phi_{m,a}(s) - \frac{1}{s-1}$ is holomorphic on $\mathrm{Re}(s) \geq 1$ and prove that

$$\Phi_{m,a}(s) = s\int_0^\infty e^{-st}\theta_{m,a}(e^t)dt.$$

(e) Show that the Laplace transform of $f(t) = \theta_{m,a}(e^t)e^{-t} - 1$ extends to a holomorphic function on $\mathrm{Re}(s) \geq 0$ and use this to prove $\theta_{m,a}(x) \sim x$.

(f) Show that (e) implies

$$\pi(x;m,a) \sim \frac{\pi(x)}{\phi(m)} \sim \frac{1}{\phi(m)}\frac{x}{\log x}.$$

Problem 5. Factoring with the analytic class number formula (49 points)

Let K be an imaginary quadratic field with discriminant $D < 0$. Recall from Problem 2 of Problem Set 7 that each ideal class in $\mathrm{cl}\,\mathcal{O}_K$ can be uniquely represented by a reduced binary quadratic form

$$f(x,y) = ax^2 + bxy + cy^2$$

which we compactly denote $f = (a,b,c)$. The coefficients a,b,c are integers with no common factor with $a > 0$ and $b^2 - 4ac = D$ (so f is integral, primitive, positive definite, and of discriminant D), and if

$$-a < b \leq a < c \qquad \text{or} \qquad 0 \leq b \leq a = c,$$

then we say that f is *reduced*, and in this case $a \leq \sqrt{|D|/3}$. Every form is *equivalent* (under the action of $\mathrm{SL}_2(\mathbb{Z})$) to a unique reduced form (a,b,c) that corresponds to an ideal $I(f) = a\mathbb{Z} + a\tau\mathbb{Z}$ of norm a in the class it represents, where

$$\tau := \frac{-b+\sqrt{D}}{2a}$$

and $\mathcal{O}_K = \mathbb{Z} + a\tau\mathbb{Z}$. Let σ be the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$. If $\mathfrak{a}$ is an ideal, then $\bar{\mathfrak{a}} := \sigma(\mathfrak{a})$ denotes its Galois conjugate.

Everything above also applies to orders $\mathcal{O} \subseteq \mathcal{O}_K$ that are not necessarily maximal, provided we restrict our attention to ideals whose norms are prime to the conductor $c := [\mathcal{O}_K : \mathcal{O}]$. We now work in this greater generality and consider binary quadratic forms of discriminant $D = c^2\,\mathrm{disc}\,\mathcal{O}_K$ and the class group $\mathrm{cl}\,\mathcal{O}$ (the group of ideals prime to the conductor modulo equivalence of principal ideals).

(a) Show that the identity element in $\operatorname{cl}\mathcal{O}$ is represented by the form $(1, 0, -D/4)$ when D is even and $(1, 1, (1 - D)/4)$ when D is odd.

(b) Show that if $\mathfrak{a}$ is an ideal with Galois conjugate $\bar{\mathfrak{a}}$ then $\mathfrak{a}\bar{\mathfrak{a}} = (N(\mathfrak{a}))$ and therefore $[\mathfrak{a}]^{-1} = [\bar{\mathfrak{a}}]$. Show that in terms of forms, if $\mathfrak{a} = I(f)$ with $f = (a, b, c)$ then $\bar{\mathfrak{a}}$ corresponds to the form $(a, -b, c)$, and if $(a, -b, c)$ is not reduced then we must have $b = a$ or $a = c$, but in both these cases $(a, -b, c)$ is equivalent to (a, b, c).

(c) An *ambiguous form* $f = (a, b, c)$ is a reduced form for which one of the following holds: $b = 0$, $b = a$, or $c = a$. Show that every ambiguous form corresponds to an ideal class that is equal to its inverse (hence has order 1 or 2), and conversely.

(d) Show that if D is odd then the ambiguous forms of discriminant D are those of the form

$$\left(\tfrac{u+v}{4}, \tfrac{v-u}{2}, \tfrac{u+v}{4}\right)$$

with $uv = -D$, $\gcd(u, v) = 1$, and $0 < v/3 \le u \le v$, and those of the form

$$\left(u, u, \tfrac{u+v}{4}\right)$$

with $uv = -D$, $\gcd(u, v) = 1$, and $0 < u \le v/3$.

(e) Show that if D is odd and has k distinct prime factors then there are 2^{k-1} ambiguous forms, each representing a 2-torsion element of $\operatorname{cl}\mathcal{O}$ (an ideal class of order 1 or 2), and conversely, that every 2-torsion element of $\operatorname{cl}\mathcal{O}$ is represented by an ambiguous form. Conclude that the 2-torsion subgroup of $\operatorname{cl}\mathcal{O}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{k-1}$ and that every ideal class of order 1 or 2 is represented by an ambiguous form.

(f) Let $n > 1$ be an integer coprime to 6, not a perfect power. Show that if $n \equiv 3 \bmod 4$ then for the discriminant $D = -n$ every ideal class in $\operatorname{cl}\mathcal{O}$ of order 2 (of which there is at least one) is represented by an ambiguous form whose coefficients yield a nontrivial factorization uv of n; show that if $n \equiv 1 \bmod 4$ then for the discriminant $D = -3n$ a similar statement holds for all but one ideal class of order 2 (of which there are at least 3).

(g) Show that for $\mathcal{O} = \mathcal{O}_K$ we have $\#\operatorname{cl}\mathcal{O} = \frac{1}{\pi}\sqrt{|D|}L(1, \chi)$, where χ is the Dirichlet character defined by the Kronecker symbol $\left(\frac{D}{\cdot}\right)$ (so $\chi(n) = \left(\frac{D}{n}\right)$). This also holds for $\mathcal{O} \subsetneq \mathcal{O}_K$, but you are not required to prove this.

The Extended Riemann Hypothesis (ERH) states that the zeros of every Dirichlet L-function $L(s, \chi)$ all lie on the critical line $\operatorname{Re}(s) = \frac{1}{2}$. Under this assumption there is an effectively computable constant c_1 such that if we compute the partial product

$$L^* := \prod_{p \le n^{1/5}} \left(1 - \chi(p)p^{-1}\right)^{-1}$$

of $L(1, \chi)$ and put $h^* := \frac{1}{\pi}\sqrt{|D|}L^*$ (with $D < -4$), then for $h = \#\operatorname{cl}\mathcal{O}$ we have

$$|h - h^*| < c_1 n^{2/5} (\log n)^2;$$

as shown in [**?**]. The ERH also implies the existence of an effectively computable constant c_2 for which the set of ideals of prime norm $a \le c_2 \log^2 |D|$ are enough to generate $\operatorname{cl}\mathcal{O}$; this follows from results in [**?**] (for $\mathcal{O} = \mathcal{O}_K$ one can take $c_2 = 6$, see [**?**]).

(h) Describe a deterministic $O(n^{1/5+o(1)})$ algorithm that, given an integer $n > 1$ does one of the following: (1) outputs a nontrivial factorization of n, (2) proves that n is prime, (3) proves that the ERH is false. Assume arithmetic operations on integers (and rational numbers) can be performed in quasi-linear time (i.e. $O(b^{1+o(1)})$ where b is the number of bits in the operands). You do not need to spell out all the details of the algorithm, a summary of each step is sufficient (note: you will need to address the case where n is a perfect power separately). If you are not familiar with the baby-steps giant-steps algorithm, see section 8.9 in these notes for a quick overview).

Problem 6. Survey (2 points)

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			
Problem 5			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
11/11	Dirichlet L-functions				
11/13	Analytic class number formula				
11/18	Kronecker-Weber theorem				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

References

[1] E. Bach, *Explicit bounds for primality testing and related problems*, Math. Comp. **55** (1990), 335–380.

[2] J.C. Lagarias, H.L. Montgomery, and A.M. Odlyzko, *A bound for the least prime ideal in the Chebotarev Density Theorem*, Invent. Math. **54** (1979), 271–296.

[3] R. Schoof, *Quadratic fields and factorization*, in "Computational Methods in Number Theory", MC-Tracts 154/155, 1982, 235–286.

[4] D. Zagier, *Newman's short proof of the prime number theorem*, Amer. Math. Monthly **104** (1997), 705–708.