10 Excision and applications

We have found two general properties of singular homology: homotopy invariance and the long exact sequence of a pair. We also claimed that $H_*(X,A)$ "depends only on $X - A$." You have to be careful about this. The following definition gives conditions that will capture the sense in which the relative homology of a pair (X,A) depends only on the complement of A in X.

Definition 10.1. A triple (X,A,U) where $U \subseteq A \subseteq X$, is *excisive* if $\overline{U} \subseteq \mathrm{Int}(A)$. The inclusion $(X-U, A-U) \subseteq (X,A)$ is then called an *excision*.

Theorem 10.2. *An excision induces an isomorphism in homology,*

$$H_*(X-U, A-U) \xrightarrow{\cong} H_*(X,A)\,.$$

So you can cut out closed bits of the interior of A without changing the relative homology. The proof will take us a couple of days. Before we give applications, let me pose a different way to interpret the motto "$H_*(X,A)$ depends only on $X-A$." Collapsing the subspace A to a point gives us a map of pairs

$$(X,A) \to (X/A,*)\,.$$

When does this map induce an isomorphism in homology? Excision has the following consequence.

Corollary 10.3. *Assume that there is a subspace B of X such that (1) $\overline{A} \subseteq \mathrm{Int}B$ and (2) $A \to B$ is a deformation retract. Then*

$$H_*(X,A) \to H_*(X/A,*)$$

is an isomorphism.

Proof. The diagram of pairs

$$\begin{array}{ccccc}
(X,A) & \xrightarrow{i} & (X,B) & \xleftarrow{j} & (X-A,B-A) \\
\downarrow & & \downarrow & & \downarrow k \\
(X/A,*) & \xrightarrow{\bar{\imath}} & (X/A,B/A) & \xleftarrow{\bar{\jmath}} & (X/A-*,B/A-*)
\end{array}$$

commutes. We want the left vertical to be a homology isomorphism, and will show that the rest of the perimeter consists of homology isomorphisms. The map k is a homeomorphism of pairs while j is an excision by assumption (1). The map i induces an isomorphism in homology by assumption (2), the long exact sequences, and the five-lemma. Since I is a compact Hausdorff space, the map $B \times I \to B/A \times I$ is again a quotient map, so the deformation $B \times I \to B$, which restricts to the constant deformation on A, descends to show that $* \to B/A$ is a deformation retract. So the map $\bar{\imath}$ is also a homology isomorphism. Finally, $\overline{*} \subseteq \mathrm{Int}(B/A)$ in X/A, by definition of the quotient topology, so $\bar{\jmath}$ induces an isomorphism by excision. □

Now what are some consequences? For a start, we'll finally get around to computing the homology of the sphere. It happens simultaneously with a computation of $H_*(D^n,S^{n-1})$. (Note that $S^{-1} = \varnothing$.) To describe generators, for each $n \geq 0$ pick a homeomorphism

$$(\Delta^n, \partial\Delta^n) \to (D^n, S^{n-1})\,,$$

and write

$$\iota_n \in S_n(D^n, S^{n-1})$$

for the corresponding relative n-chain.

Proposition 10.4. *Let $n > 0$ and let $* \in S^{n-1}$ be any point. Then:*

$$H_q(S^n) = \begin{cases} \mathbf{Z} = \langle [\partial\iota_{n+1}] \rangle & \textit{if} \quad q = n > 0 \\ \mathbf{Z} = \langle [c_*^0] \rangle & \textit{if} \quad q = 0, n > 0 \\ \mathbf{Z} \oplus \mathbf{Z} = \langle [c_*^0], [\partial\iota_1] \rangle & \textit{if} \quad q = n = 0 \\ 0 & \textit{otherwise} \end{cases}$$

and

$$H_q(D^n, S^{n-1}) = \begin{cases} \mathbf{Z} = \langle [\iota_n] \rangle & \textit{if} \quad q = n \\ 0 & \textit{otherwise}\,. \end{cases}$$

Proof. The division into cases for $H_q(S^n)$ can be eased by employing reduced homology. Then the claim is merely that for $n \geq 0$

$$\widetilde{H}_q(S^{n-1}) = \begin{cases} \mathbf{Z} & \text{if} \quad q = n-1 \\ 0 & \text{if} \quad q \neq n-1 \end{cases}$$

and the map

$$\partial : H_q(D^n, S^{n-1}) \to \widetilde{H}_{q-1}(S^{n-1})$$

is an isomorphism. The second statement follows from the long exact sequence in reduced homology together with the fact that $\widetilde{H}_*(D^n) = 0$ since D^n is contractible. The first uses induction and the pair of isomorphisms

$$\widetilde{H}_{q-1}(S^{n-1}) \xleftarrow{\cong} H_q(D^n, S^{n-1}) \xrightarrow{\cong} H_q(D^n/S^{n-1}, *)$$

since $D^n/S^{n-1} \cong S^n$. The right hand arrow is an isomorphism since S^{n-1} is a deformation retract of a neighborhood in D^n. □

Why should you care about this complicated homology calculation?

Corollary 10.5. *If $m \neq n$, then S^m and S^n are not homotopy equivalent.*

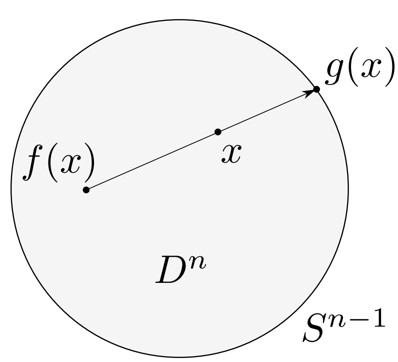
Proof. Their homology groups are not isomorphic. □

Corollary 10.6. *If $m \neq n$, then $\mathbf{R}^m$ and $\mathbf{R}^n$ are not homeomorphic.*

Proof. If m or n is zero, this is clear, so let $m, n > 0$. Assume we have a homeomorphism $f : \mathbf{R}^m \to \mathbf{R}^n$. This restricts to a homeomorphism $\mathbf{R}^m - \{0\} \to \mathbf{R}^n - \{f(0)\}$. But these spaces are homotopy equivalent to spheres of different dimension. □

Theorem 10.7 (Brouwer fixed-point theorem)**.** *If $f : D^n \to D^n$ is continuous, then there is some point $x \in D^n$ such that $f(x) = x$.*

Proof. Suppose not. Then you can draw a ray from $f(x)$ through x. It meets the boundary of D^n at a point $g(x) \in S^{n-1}$. Check that $g : D^n \to S^{n-1}$ is continuous. If x is on the boundary, then $x = g(x)$, so g provides a factorization of the identity map on S^{n-1} through D^n. This is inconsistent with our computation because the identity map induces the identity map on $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbf{Z}$, while $\widetilde{H}_{n-1}(D^n) = 0$. □



Our computation of the homology of a sphere also implies that there are many non-homotopic self-maps of S^n, for any $n \geq 1$. We will distinguish them by means of the "degree": A map $f : S^n \to S^n$ induces an endomorphism of the infinite cyclic group $H_n(S^n)$. Any endomorphism of an infinite cyclic group is given by multiplication by an integer. This integer is well defined (independent of a choice of basis), and any integer occurs. Thus $\mathrm{End}(\mathbf{Z}) = \mathbf{Z}_\times$, the monoid of integers under multiplication. The homotopy classes of self-maps of S^n also form a monoid, under composition, and:

Theorem 10.8. *Let $n \geq 1$. The degree map provides us with a surjective monoid homomorphism*

$$\deg : [S^n, S^n] \to \mathbf{Z}_\times \,.$$

Proof. Degree is multiplicative by functoriality of homology.

We construct a map of degree k on S^n by induction on n. If $n = 1$, this is just the winding number; an example is given by regarding S^1 as unit complex numbers and sending z to z^k. The proof that this has degree k is an exercise.

Suppose we've constructed a map $f_k : S^{n-1} \to S^{n-1}$ of degree k. Extend it to a map $\overline{f}_k : D^n \to D^n$ by defining $\overline{f}_k(tx) = tf_k(x)$ for $t \in [0, 1]$. We may then collapse the sphere to a point and identify the quotient with S^n. This gives us a new map $g_k : S^n \to S^n$ making the diagram below commute.

$$\begin{array}{ccccc}
H_{n-1}(S^{n-1}) & \xleftarrow{\cong} & H_n(D^n, S^{n-1}) & \xrightarrow{\cong} & H_n(S^n) \\
\downarrow{\scriptstyle f_{k*}} & & \downarrow & & \downarrow{\scriptstyle g_{k*}} \\
H_{n-1}(S^{n-1}) & \xleftarrow{\cong} & H_n(D^n, S^{n-1}) & \xrightarrow{\cong} & H_n(S^n)
\end{array}$$

The horizontal maps are isomorphisms, so $\deg g_k = k$ as well. □

We will see (in 18.906) that this map is in fact an isomorphism.