

11 The Eilenberg Steenrod axioms and the locality principle

Before we proceed to prove the excision theorem, let's review the properties of singular homology as we have developed them. They are captured by a set of axioms, due to Sammy Eilenberg and Norman Steenrod [5].

Definition 11.1. A *homology theory* (on \mathbf{Top}) is:

- a sequence of functors $h_n : \mathbf{Top}_2 \rightarrow \mathbf{Ab}$ for all $n \in \mathbf{Z}$ and
- a sequence of natural transformations $\partial : h_n(X, A) \rightarrow h_{n-1}(A, \emptyset)$

such that:

- If $f_0, f_1 : (X, A) \rightarrow (Y, B)$ are homotopic, then $f_{0*} = f_{1*} : h_n(X, A) \rightarrow h_n(Y, B)$.
- Excisions induce isomorphisms.
- For any pair (X, A) , the sequence

$$\cdots \rightarrow h_{q+1}(X, A) \xrightarrow{\partial} h_q(A) \rightarrow h_q(X) \rightarrow h_q(X, A) \xrightarrow{\partial} \cdots$$

is exact, where we have written $h_q(X)$ for $h_q(X, \emptyset)$.

- (The dimension axiom): The group $h_n(*)$ is nonzero only for $n = 0$.

We add the following “Milnor axiom” [8] to our definition. To state it, let I be a set and suppose that for each $i \in I$ we have a space X_i . We can form their disjoint union or *coproduct* $\coprod X_i$. The inclusion maps $X_i \rightarrow \coprod X_i$ induce maps $h_n(X_i) \rightarrow h_n(\coprod X_i)$, and these in turn induce a map from the direct sum, or coproduct in **Ab**:

$$\alpha : \bigoplus_{i \in I} h_n(X_i) \rightarrow h_n\left(\coprod_{i \in I} X_i\right).$$

Then:

- The map α is an isomorphism for all n .

Ordinary singular homology satisfies these, with $h_0(*) = \mathbf{Z}$. We will soon add “coefficients” to homology, producing a homology theory whose value on a point is any prescribed abelian group. In later developments, it emerges that the dimension axiom is rather like the parallel postulate in Euclidean geometry: it’s “obvious,” but, as it turns out, the remaining axioms accomodate extremely interesting alternatives, in which $h_n(*)$ is nonzero for infinitely many values of n (both positive and negative).

Excision is a statement that homology is “localizable.” To make this precise, we need some definitions.

Definition 11.2. Let X be a topological space. A family \mathcal{A} of subsets of X is a *cover* if X is the union of the interiors of elements of \mathcal{A} .

Definition 11.3. Let \mathcal{A} be a cover of X . An n -simplex σ is *\mathcal{A} -small* if there is $A \in \mathcal{A}$ such that the image of σ is entirely in A .

Notice that if $\sigma : \Delta^n \rightarrow X$ is \mathcal{A} -small, then so is $d_i\sigma$; in fact, for any simplicial operator ϕ , $\phi^*\sigma$ is again \mathcal{A} -small. Let’s denote by $\text{Sin}_*^{\mathcal{A}}(X)$ the graded set of \mathcal{A} -small simplices. This is a sub-simplicial set of $\text{Sin}_*(X)$. Applying the free abelian group functor, we get the subcomplex

$$S_*^{\mathcal{A}}(X)$$

of *\mathcal{A} -small singular chains*. Write $H_*^{\mathcal{A}}(X)$ for its homology.

Theorem 11.4 (The locality principle). *The inclusion $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ induces an isomorphism in homology, $H_*^{\mathcal{A}}(X) \xrightarrow{\cong} H_*(X)$.*

This will take a little time to prove. Let’s see right now how it implies excision.

Suppose $X \supset A \supset U$ is excisive, so that $\bar{U} \subseteq \text{Int}A$, or $\text{Int}(X - U) \cup \text{Int}A = X$. This if we let $B = X - U$, then $\mathcal{A} = \{A, B\}$ is a cover of X . Rewriting in terms of B ,

$$(X - U, A - U) = (B, A \cap B),$$

so we aim to show that

$$S_*(B, A \cap B) \rightarrow S_*(X, A)$$

induces an isomorphism in homology. We have the following diagram of chain complexes with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S_*(A) & \longrightarrow & S_*^A(X) & \longrightarrow & S_*^A(X)/S_*(A) \longrightarrow 0 \\
 & & \downarrow = & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S_*(A) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X, A) \longrightarrow 0
 \end{array}$$

The middle vertical induces an isomorphism in homology by the locality principle, so the homology long exact sequences combine with the five-lemma to show that the right hand vertical is also a homology isomorphism. But

$$S_n^A(X) = S_n(A) + S_n(B) \subseteq S_n(X)$$

and a simple result about abelian groups provides an isomorphism

$$\frac{S_n(B)}{S_n(A \cap B)} = \frac{S_n(B)}{S_n(A) \cap S_n(B)} \cong \frac{S_n(A) + S_n(B)}{S_n(A)} = \frac{S_n^A(X)}{S_n(A)},$$

so excision follows.

This case of a cover with two elements leads to another expression of excision, known as the ‘‘Mayer-Vietoris sequence.’’ In describing it we will use the following notation for the various inclusion.

$$\begin{array}{ccc}
 A \cap B & \xrightarrow{j_1} & A \\
 \downarrow j_2 & & \downarrow i_1 \\
 B & \xrightarrow{i_2} & X
 \end{array}$$

Theorem 11.5 (Mayer-Vietoris). *Assume that $\mathcal{A} = \{A, B\}$ is a cover of X . There are natural maps $\partial : H_n(X) \rightarrow H_{n-1}(A \cap B)$ such that the sequence*

$$\begin{array}{ccccc}
 & & \cdots & \xrightarrow{\beta} & H_{n+1}(X) \\
 & & & \searrow \partial & \\
 H_n(A \cap B) & \xleftarrow{\alpha} & H_n(A) \oplus H_n(B) & \xrightarrow{\beta} & H_n(X) \\
 & & & \searrow \partial & \\
 H_{n-1}(A \cap B) & \xleftarrow{\alpha} & \cdots & &
 \end{array}$$

is exact, where

$$\alpha = \begin{bmatrix} j_{1*} \\ -j_{2*} \end{bmatrix}, \quad \beta = [i_{1*} \quad i_{2*}].$$

Proof. This is the homology long exact sequence associated to the short exact sequence of chain complexes

$$0 \rightarrow S_*(A \cap B) \xrightarrow{\alpha} S_*(A) \oplus S_*(B) \xrightarrow{\beta} S_*^A(X) \rightarrow 0,$$

combined with the locality principle. □

The Mayer-Vietoris theorem follows from excision as well, via the following simple observation. Suppose we have a map of long exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{k} & A'_n & \longrightarrow & B'_n & \longrightarrow & C'_n & \longrightarrow & \cdots \\ & & \downarrow h & & \downarrow f & & \downarrow \cong & & \downarrow h & & \\ \cdots & \longrightarrow & C_{n+1} & \xrightarrow{k} & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & \cdots \end{array}$$

in which every third arrow is an isomorphism as indicated. Define a map

$$\partial : A_n \rightarrow B_n \xleftarrow{\cong} B'_n \rightarrow C'_n.$$

An easy diagram chase shows:

Lemma 11.6. *The sequence*

$$\cdots \longrightarrow C'_{n+1} \xrightarrow{\begin{bmatrix} h \\ -k \end{bmatrix}} C_{n+1} \oplus A'_n \xrightarrow{\begin{bmatrix} k & f \end{bmatrix}} A_n \xrightarrow{\partial} C'_n \longrightarrow \cdots$$

is exact.

To get the Mayer-Vietoris sequence, let $\{A, B\}$ be a cover of X and apply the lemma to

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & H_n(A \cap B) & \longrightarrow & H_n(B) & \longrightarrow & H_n(B, A \cap B) & \longrightarrow & H_{n-1}(A \cap B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow & \cdots \end{array}$$

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