Chapter 2

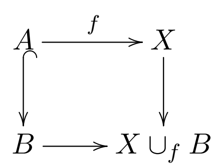
Computational methods

14 CW-complexes

There are various ways to model geometrically interesting spaces. Manifolds provide one important model, well suited to analysis. Another model, one we have not talked about, is given by simplicial complexes. It's very combinatorial, and constructing a simplicial complex model for a given space involves making a lot of choices that are combinatorial rather than topological in character. A more flexible model, one more closely reflecting topological information, is given by the theory of CW-complexes.

In building up a space as a CW-complex, we will successively "glue" cells onto what has been already built. This is a general construction.

Suppose we have a pair (B, A), and a map $f : A \to X$. Define a space $X \cup_f B$ (or $X \cup_A B$) in the diagram

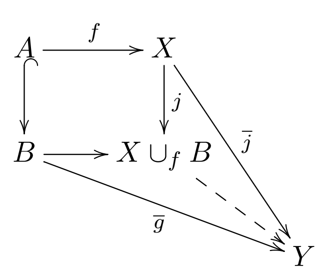


by

$$X \cup_f B = X \sqcup B / \sim$$

where the equivalence relation is generated by requiring that $a \sim f(a)$ for all $a \in A$. We say that we have "attached B to X along f (or along A)."

There are two kinds of equivalence classes in $X \cup_f B$: (1) singletons containing elements of $B - A$, and (2) $\{x\} \sqcup f^{-1}(x)$ for $x \in X$. The topology on $X \cup_f B$ is the quotient topology, and is characterized by a universal property: any solid-arrow commutative diagram



can be uniquely filled in. It's a "push-out."

Example 14.1. If $X = *$, then $* \cup_f B = B/A$.

Example 14.2. If $A = \varnothing$, then $X \cup_f B$ is the coproduct $X \sqcup B$.

Example 14.3. If both,

$$B/\varnothing = * \cup_\varnothing B = * \sqcup B\,.$$

For example, $\varnothing/\varnothing = *$. This is creation from nothing. We won't get into the religious ramifications.

Example 14.4 (Attaching a cell)**.** A basic collection of pairs of spaces is given by the disks relative to their boundaries: (D^n, S^{n-1}). (Recall that $S^{-1} = \varnothing$.) In this context, D^n is called an "n-cell," and a map $f : S^{n-1} \to X$ allows us to attach an n-cell to X, to form

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & X \\ \big\downarrow & & \big\downarrow \\ D^n & \longrightarrow & X \cup_f D^n \end{array}$$

You might want to generalize this a little bit, and attach a bunch of n-cells all at once:

$$\begin{array}{ccc} \coprod_{\alpha\in A} S^{n-1}_\alpha & \xrightarrow{f} & X \\ \big\downarrow & & \big\downarrow \\ \coprod_{\alpha\in A} D^n_\alpha & \longrightarrow & X \cup_f \coprod_{\alpha\in A} D^n_\alpha \end{array}$$

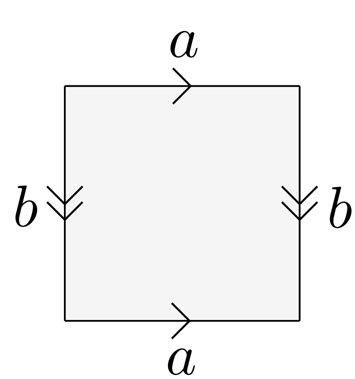
What are some examples? When $n = 0$, $(D^0, S^{-1}) = (*, \varnothing)$, so you are just adding a discrete set to X:

$$X \cup_f \coprod_{\alpha\in A} D^0 = X \sqcup A$$

More interesting: Let's attach two 1-cells to a point:

$$\begin{array}{ccc} S^0 \sqcup S^0 & \xrightarrow{f} & * \\ \big\downarrow & & \big\downarrow \\ D^1 \sqcup D^1 & \longrightarrow & * \cup_f (D^1 \sqcup D^1) \end{array}$$

Again there's just one choice for f, and $* \cup_f (D^1 \sqcup D^1)$ is a figure 8, because you start with two 1-disks and identify the four boundary points together. Let me write $S^1 \vee S^1$ for this space. We can go on and attach a single 2-cell to manufacture a torus. Think of the figure 8 as the perimeter of a square with opposite sides identified.



The inside of the square is a 2-cell, attached to the perimeter by a map I'll denote by $aba^{-1}b^{-1}$:

$$\begin{array}{ccc} S^1 & \xrightarrow{aba^{-1}b^{-1}} & S^1 \vee S^1 \\ \big\downarrow & & \big\downarrow \\ D^2 & \longrightarrow & (S^1 \vee S^1) \cup_f D^2 = T^2 . \end{array}$$

This example illuminates the following definition.

Definition 14.5. A *CW-complex* is a space X equipped with a sequence of subspaces

$$\varnothing = \mathrm{Sk}_{-1}X \subseteq \mathrm{Sk}_0 X \subseteq \mathrm{Sk}_1 X \subseteq \cdots \subseteq X$$

such that

- X is the union of the $\mathrm{Sk}_n X$'s, and
- for all n, there is a pushout diagram like this:

$$\begin{array}{ccc} \coprod_{\alpha \in A_n} S^{n-1}_\alpha & \xrightarrow{f_n} & \mathrm{Sk}_{n-1}X \\ \big\downarrow & & \big\downarrow \\ \coprod_{\alpha \in A_n} D^n_\alpha & \xrightarrow{g_n} & \mathrm{Sk}_n X \end{array} .$$

The subspace $\mathrm{Sk}_n X$ is the *n-skeleton* of X. Sometimes it's convenent to use the alternate notation X_n for the n-skeleton. The first condition is intended topologically, so that a subset of X is open if and only if its intersection with each $\mathrm{Sk}_n X$ is open; or, equivalently, a map $f : X \to Y$ is continuous if and only if its restriction to each $\mathrm{Sk}_n X$ is continuous. The maps f_n are the *attaching maps* and the maps g_n are *characteristic maps.*

Example 14.6. We just constructed the torus as a CW complex with $\mathrm{Sk}_0 T^2 = *$, $\mathrm{Sk}_1 T^2 = S^1 \vee S^1$, and $\mathrm{Sk}_2 T^2 = T^2$.

Definition 14.7. A CW-complex is *finite-dimensional* if $\mathrm{Sk}_n X = X$ for some n; of *finite type* if each A_n is finite, i.e., finitely many cell in each dimension; and *finite* if it's finite-dimensional and of finite type.

The *dimension* of a CW complex is the largest n for which there are n-cells. This is not obviously a topological invariant, but, have no fear, it turns out that it is.

In "CW," the "C" is for cell, and the "W" is for weak, because of the topology on a CW-complex. This definition is due to J. H. C. Whitehead. Here are a couple of important facts about them.

Theorem 14.8. *Any CW-complex is Hausdorff, and it's compact if and only if it's finite.*
Any compact smooth manifold admits a CW structure.

Proof. See [2] Prop. IV.8.1, [6] Prop. A.3. □

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