15 CW-complexes II

We have a few more general things to say about CW complexes.

Suppose X is a CW complex, with skeleton filtration $\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ and cell structure

$$\begin{array}{ccc} \coprod_{\alpha\in A_n} S^{n-1}_\alpha & \xrightarrow{f_n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\alpha\in A_n} D^n_\alpha & \xrightarrow{g_n} & X_n \end{array} .$$

In each case, the boundary of a cell gets identified with part of the previous skeleton, but the "interior"

$$\mathrm{Int}D^n = \{x \in D^n : |x| < 1\}$$

does not. (Note that $\mathrm{Int}D^0 = D^0$.) Thus as sets – ignoring the topology –

$$X = \coprod_{n\geq 0} \coprod_{\alpha\in A_n} \mathrm{Int}(D^n_\alpha) .$$

The subsets $\mathrm{Int}D^n_\alpha$ are called "open n-cells," despite the fact that they not generally open in the topology on X, and (except when $n = 0$) they are not homeomorphic to compact disks.

Definition 15.1. Let X be a CW-complex with a cell structure $\{g_\alpha : D^n_\alpha \to X_n : \alpha \in A_n, n \geq 0\}$. A *subcomplex* is a subspace $Y \subseteq X$ such that for all n, there is a subset B_n of A_n such that $Y_n = Y \cap X_n$ provides Y with a CW-structure with characteristic maps $\{g_\beta : \beta \in B_n, n \geq 0\}$.

Example 15.2. $\mathrm{Sk}_n X \subseteq X$ is a subcomplex.

Proposition 15.3. *Let X be a CW-complex with a chosen cell structure. Any compact subspace of X lies in some finite subcomplex.*

Proof. See [2], p. 196. □

Remark 15.4. For fixed cell structures, unions and intersections of subcomplexes are subcomplexes.

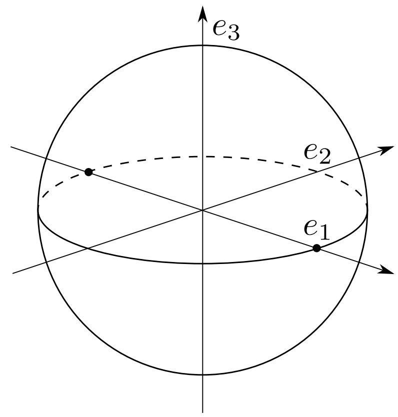
The n-sphere S^n (for $n > 0$) admits a very simple CW structure: Let $* = \mathrm{Sk}_0(S^n) = \mathrm{Sk}_1(S^n) = \cdots = \mathrm{Sk}_{n-1}(S^n)$, and attach an n-cell using the unique map $S^{n-1} \to *$. This is a minimal CW structure – you need at least two cells to build S^n.

This is great – much simpler than the simplest construction of S^n as a simplicial complex – but it is not ideal for all applications. Here's another CW-structure on S^n. Regard $S^n \subseteq \mathbf{R}^{n+1}$, filter the Euclidean space by leading subspaces

$$\mathbf{R}^k = \langle e_1, \ldots, e_k\rangle .$$

and define

$$\mathrm{Sk}_k S^n = S^n \cap \mathbf{R}^{k+1} = S^k .$$



Now there are two k-cells for each k with $0 \leq k \leq n$, given by the two hemispheres of S^k. For each k there are two characteristic maps,

$$u, \ell : D^k \to S^k$$

defining the upper and lower hemispheres:

$$u(x) = (x, \sqrt{1 - |x|^2}), \quad \ell(x) = (x, -\sqrt{1 - |x|^2}).$$

Note that if $|x| = 1$ then $|u(x)| = |\ell(x)| = 1$, so each characteristic map restricts on the boundary to a map to S^{k-1}, and serves as an attaching map. This cell structure has the advantage that S^{n-1} is a subcomplex of S^n.

The case $n = \infty$ is allowed here. Then $\mathbf{R}^\infty$ denotes the countably infinite dimensional inner product space that is the topological union of the leading subspaces $\mathbf{R}^n$. The CW-complex S^∞ is of finite type but not finite dimensional. It has the following interesting property. We know that S^n is not contractible (because the identity map and a constant map have different behavior in homology), but:

Proposition 15.5. *S^∞ is contractible.*

Proof. This is an example of a "swindle," making use of infinite dimensionality. Let $T : \mathbf{R}^\infty \to \mathbf{R}^\infty$ send $(x_1, x_2, \ldots)$ to $(0, x_1, x_2, \ldots)$. This sends S^∞ to itself. The location of the leading nonzero entry is different for x and Tx, so the line segment joining x to Tx doesn't pass through the origin. Therefore

$$x \mapsto \frac{tx + (1-t)Tx}{|tx + (1-t)Tx|}$$

provides a homotopy $1 \simeq T$. On the other hand, T is homotopic to the constant map with value $(1, 0, 0, \ldots)$, again by an affine homotopy. □

This "inefficient" CW structure on S^n has a second advantage: it's *equivariant* with respect to the antipodal involution. This provides us with a CW structure on the orbit space for this action.

Recall that $\mathbf{RP}^k = S^k/\sim$ where $x \sim -x$. The quotient map $\pi : S^k \to \mathbf{RP}^k$ is a double cover, identifying upper and lower hemispheres. The inclusion of one sphere in the next is compatible with this equivalence relation, and gives us "linear" embeddings $\mathbf{RP}^{k-1} \subseteq \mathbf{RP}^k$. This suggests that

$$\varnothing \subseteq \mathbf{RP}^0 \subseteq \mathbf{RP}^1 \subseteq \cdots \subseteq \mathbf{RP}^n$$

might serve as a CW filtration. Indeed, for each k,

$$\begin{array}{ccc} S^{k-1} & \longrightarrow & D^k \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle u} \\ \mathbf{RP}^{k-1} & \longrightarrow & \mathbf{RP}^k \end{array}$$

is a pushout: A line in $\mathbf{R}^{k+1}$ either lies in $\mathbf{R}^k$ or is determined by a unique point in the upper hemisphere of S^k.

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18.905 Algebraic Topology I
Fall 2016