16 Homology of CW-complexes

The skeleton filtration of a CW complex leads to a long exact sequence in homology, showing that the relative homology $H_*(X_k, X_{k-1})$ controls how the homology changes when you pass from X_{k-1} to X_k. What is this relative homology? If we pick a set of attaching maps, we get the following diagram.

$$\begin{array}{ccccc} \coprod_\alpha S^{k-1} & \hookrightarrow & \coprod_\alpha D^k_\alpha & \longrightarrow & \bigvee_\alpha S^k_\alpha \\ \Big\downarrow f & & \Big\downarrow & & \vdots \\ X_{k-1} & \hookrightarrow & X_k \cup_f B & \longrightarrow & X_k/X_{k-1} \end{array}$$

where $\bigvee$ is the wedge sum (disjoint union with all basepoints identified): $\bigvee_\alpha S^k_\alpha$ is a bouquet of spheres. The dotted map exists and is easily seen to be a homeomorphism.

Luckily, the inclusion $X_{k-1} \subseteq X_k$ satisfies what's needed to conclude that

$$H_q(X_k, X_{k-1}) \to H_q(X_k/X_{k-1}, *)$$

is an isomorphism. After all, X_{k-1} is a deformation retract of the space you get from X_k by deleting the center of each k-cell.

We know $H_q(X_k/X_{k-1}, *)$ very well:

$$H_q(\bigvee_{\alpha \in A_k} S^k_\alpha, *) \cong \begin{cases} \mathbf{Z}[A_k] & q = k \\ 0 & q \neq k \end{cases} .$$

Lesson: The relative homology $H_k(X_k, X_{k-1})$ keeps track of the k-cells of X.

Definition 16.1. The group of *cellular n-chains* in a CW complex X is

$$C_k(X) := H_k(X_k, X_{k-1}) = \mathbf{Z}[A_k] .$$

If we put the fact that $H_q(X_k, X_{k-1}) = 0$ for $q \neq k, k+1$ into the homology long exact sequence of the pair, we find first that

$$H_q(X_{k-1}) \xrightarrow{\cong} H_q(X_k) \quad \text{for} \quad q \neq k, k-1 ,$$

and then that there is a short exact sequence

$$0 \to H_k(X_k) \to C_k(X) \to H_{k-1}(X_{k-1}) \to 0 .$$

So if we fix a dimension q, and watch how H_q varies as we move through the skelata of X, we find the following picture. Say $q > 0$. Since X_0 is discrete, $H_q(X_0) = 0$. Then $H_q(X_k)$ continues to be 0 till you get up to X_q. $H_q(X_q)$ is a subgroup of the free abelian group $C_q(X)$ and hence is free abelian. Relations may get introduced into it when we pass to X_{q+1}; but thereafter all the maps

$$H_q(X_{q+1}) \to H_q(X_{q+2}) \to \cdots$$

are isomorphisms. All the q-dimensional homology of X is created on X_q, and all the relations in $H_q(X)$ occur by X_{q+1}.

This stable value of $H_q(X_k)$ maps isomorphically to $H_q(X)$, even if X is infinite dimensional. This is because the union of the images of any finite set of singular simplices in X is compact and so lies in a finite subcomplex and in particular lies in a finite skeleton. So any chain in X is the image of a chain in some skeleton. Since $H_q(X_k) \xrightarrow{\cong} H_q(X_{k+1})$ for $k > q$, we find that $H_q(X_q) \to H_q(X)$ is surjective. Similarly, if $c \in S_q(X_k)$ is a boundary in X, then it's a boundary in X_ℓ for some $\ell \geq k$. This shows that the map $H_q(X_{q+1}) \to H_q(X)$ is injective. We summarize:

Proposition 16.2. *Let $k, q \geq 0$. Then*

$$H_q(X_k) = 0 \quad \text{for } k < q$$

and

$$H_q(X_k) \xrightarrow{\cong} H_q(X) \quad \text{for } k > q\,.$$

In particular, $H_q(X) = 0$ if q exceeds the dimension of X.

We have defined the cellular n-chains of a CW complex X,

$$C_n(X) = H_n(X_n, X_{n-1})\,,$$

and found that it is the free abelian group on the set of n cells. We claim that these abelian groups are related to each other; they form the groups in a chain complex.

What should the boundary of an n-cell be? It's represented by a characteristic map $D^n \to X_n$ whose boundary is the attaching map $\alpha : S^{n-1} \to X_{n-1}$. This is a lot of information, and hard to interpret because X_{n-1} is itself potentially a complicated space. But things get much simpler if I pinch out X_{n-2}. This suggests defining

$$d : C_n(X) = H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_n) \to H_{n-1}(X_{n-1}, X_{n-2}) = C_{n-1}(X)\,.$$

The fact that $d^2 = 0$ is embedded in the following large diagram, in which the two columns and the central row are exact.

$$\begin{array}{ccccc}
C_{n+1}(X) = H_{n+1}(X_{n+1}, X_n) & & & & 0 = H_{n-1}(X_{n-2}) \\
\downarrow \partial_n & \searrow^{d} & & & \downarrow \\
H_n(X_n) & \xrightarrow{j_n} & C_n(X) = H_n(X_n, X_{n-1}) & \xrightarrow{\partial_{n-1}} & H_{n-1}(X_{n-1}) \\
\downarrow & & & \searrow^{d} & \downarrow j_{n-1} \\
H_n(X_{n+1}) & & & & C_{n-1}(X) = H_{n-1}(X_{n-1}, X_{n-2}) \\
\downarrow & & & & \\
0 = H_n(X_{n+1}, X_n) & & & &
\end{array}$$

Now, $\partial_{n-1} \circ j_n = 0$. So the composite of the diagonals is zero, i.e., $d^2 = 0$, and we have a chain complex! This is the "cellular chain complex" of X.

We should compute the homology of this chain complex, $H_n(C_*(X)) = \ker d/\operatorname{im} d$. Now

$$\ker d = \ker(j_{n-1} \circ \partial_{n-1}) .$$

But j_{n-1} is injective, so

$$\ker d = \ker \partial_{n-1} = \operatorname{im} j_n = H_n(X_n) .$$

On the other hand

$$\operatorname{im} d = j_n(\operatorname{im} \partial_n) = \operatorname{im} \partial_n \subseteq H_n(X_n) .$$

So

$$H_n(C_*(X)) = H_n(X_n)/\operatorname{im} \partial_n = H_n(X_{n+1})$$

by exactness of the left column; but as we know this is exactly $H_n(X)$! We have proven the following result.

Theorem 16.3. *For a CW complex X, there is an isomorpphism*

$$H_*(C_*(X)) \cong H_*(X)$$

natural with respect to filtration-preserving maps between CW complexes.

This has an immediate and surprisingly useful corollary.

Corollary 16.4. *Suppose that the CW complex X has only even cells – that is, $X_{2k} \hookrightarrow X_{2k+1}$ is an isomorphism for all k. Then*

$$H_*(X) \cong C_*(X) .$$

That is, $H_n(X) = 0$ for n odd, is free abelian for all n, and the rank of $H_n(X)$ for n even is the number of n-cells.

Example 16.5. Complex projective space $\mathbf{CP}^n$ has a CW structure in which

$$\mathrm{Sk}_{2k}\mathbf{CP}^n = \mathrm{Sk}_{2k+1}\mathbf{CP}^n = \mathbf{CP}^k .$$

The attaching $S^{2k-1} \to \mathbf{CP}^k$ sends $v \in S^{2k-1} \subseteq \mathbf{C}^n$ to the complex line through v. So

$$H_k(\mathbf{CP}^n) = \begin{cases} \mathbf{Z} & \text{for } 0 \leq k \leq 2n,\ k \text{ even} \\ 0 & \text{otherwise} . \end{cases}$$

Finally, notice that in our proof of Theorem 16.3 we used only properties contained in the Eilenberg-Steenrod axioms. As a result, any construction of a homology theory satisfying the Eilenberg-Steenrod axioms gives you the same values on CW complexes as singular homology.

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