19 Coefficients

Abelian groups can be quite complicated, even finitely generated ones. Vector spaces over a field are so much simpler! A vector space is determined up to isomorphism by a single cardinality, its dimension. Wouldn't it be great to have a version of homology that took values in the category of vector spaces over a field?

We can do this, and more. Let R be any commutative ring at all. Instead of forming the free abelian group on $\mathrm{Sin}_*(X)$, we could just as well form the free R-module:

$$S_*(X;R) = R\mathrm{Sin}_*(X)$$

This gives, first, a simplicial object in the category of R-modules. Forming the alternating sum of the face maps produces a chain complex *of R-modules*: $S_n(X;R)$ is an R-module for each n, and $d : S_n(X;R) \to S_{n-1}(X;R)$ is an R-module homomorphism. The homology groups are then again R-modules:

$$H_n(X;R) = \frac{\ker(d : S_n(X;R) \to S_{n-1}(X;R))}{\mathrm{im}(d : S_{n+1}(X;R) \to S_n(X;R))}\,.$$

This is the *singular homology of X with coefficients in the commutative ring R*. It satisfies all the Eilenberg-Steenrod axioms, with

$$H_n(*;R) = \begin{cases} R & \text{for} \quad n = 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

(We could actually have replaced the ring R by any abelian group here, but this will become much clearer after we have the tensor product as a tool.) This means that all the work we have done for "integral homology" carries over to homology with any coefficients. In particular, if X is a

CW complex we have the cellular homology with coefficients in R, $C_*(X;R)$, and its homology is isomorphic to $H_*(X;R)$.

The coefficient rings that are most important in algebraic topology are simple ones: the integers and the prime fields $\mathbf{F}_p$ and $\mathbf{Q}$; almost always a PID.

As an experiment, let's compute $H_*(\mathbf{RP}^n;R)$ for various rings R. Let's start with $R=\mathbf{F}_2$, the field with 2 elements. This is a favorite among algebraic topologists, because using it for coefficients eliminates all sign issues. The cellular chain complex has $C_k(\mathbf{RP}^n;\mathbf{F}_2)=\mathbf{F}_2$ for $0\le k\le n$, and the differential alternates between multiplication by 2 and by 0. But in $\mathbf{F}_2$, $2=0$: so $d=0$, and the cellular chains coincide with the homology:

$$H_k(\mathbf{RP}^n;\mathbf{F}_2)=\begin{cases}\mathbf{F}_2 & \text{for}\quad 0\le k\le n\\ 0 & \text{otherwise}\,.\end{cases}$$

On the other hand, suppose that R is a ring in which 2 is invertible. The universal case is $\mathbf{Z}[1/2]$, but any subring of the rationals containing $1/2$ would do just as well, as would $\mathbf{F}_p$ for p odd. Now the cellular chain complex (in dimensions 0 through n) looks like

$$R\xleftarrow{0}R\xleftarrow{\cong}R\xleftarrow{0}R\xleftarrow{\cong}\cdots\xleftarrow{\cong}R$$

for n even, and

$$R\xleftarrow{0}R\xleftarrow{\cong}R\xleftarrow{0}R\xleftarrow{\cong}\cdots\xleftarrow{0}R$$

for n odd. Therefore for n even

$$H_k(\mathbf{RP}^n;R)=\begin{cases}R & \text{for}\quad k=0\\ 0 & \text{otherwise}\end{cases}$$

and for n odd

$$H_k(\mathbf{RP}^n;R)=\begin{cases}R & \text{for}\quad k=0\\ R & \text{for}\quad k=n\\ 0 & \text{otherwise}\,.\end{cases}$$

You get a much simpler result: Away from 2, even projective spaces look like points, and odd projective spaces look like spheres!

I'd like to generalize this process a little bit, and allow coefficients not just in a commutative ring, but more generally in a module M over a commutative ring; in particular, any abelian group. This is most cleanly done using the mechanism of the tensor product. That mechanism will also let us address the following natural question:

Question 19.1. Given $H_*(X;R)$, can we deduce $H_*(X;M)$ for an R-module M?

The answer is called the "universal coefficient theorem". I'll spend a few days developing what we need to talk about this.

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