21 Tensor and Tor

We continue to study properties of the tensor product. Recall that

$$A \otimes \mathbf{Z}/n\mathbf{Z} = A/nA\,.$$

Consider the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0\,.$$

Let's tensor it with $\mathbf{Z}/2\mathbf{Z}$. We get

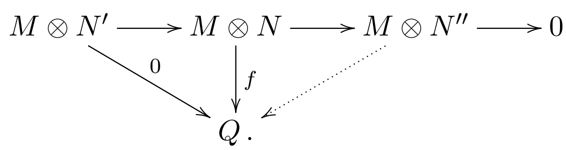
$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0\,.$$

This cannot be a short exact sequence! This is a major tragedy: tensoring doesn't preserve exact sequences; one says that the functor $\mathbf{Z}/n\mathbf{Z} \otimes -$ is not "exact." This is why we can't form homology with coefficients in M by simply tensoring homology with M.

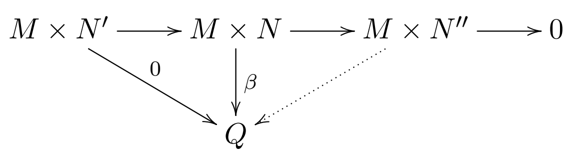
Tensoring does respect certain exact sequences:

Proposition 21.1. *The functor $N \mapsto M \otimes_R N$ preserves cokernels; it is* right exact.

Proof. Suppose that $N' \to N \to N'' \to 0$ is exact and let $f : M \otimes N \to Q$. We wish to show that there is a unique factorization as shown in the diagram



This is equivalent to asking whether there is a unique factorization of the corresponding diagram of bilinear maps,



– uniqueness of the linear factorization is guaranteed by the fact that $M \times N''$ generates $M \otimes N''$. This unique factorization reflects the fact that $M \times -$ preserves cokernels. □

Failure of exactness is bad, so let's try to repair it. A key observation is that if M is *free*, then $M \otimes_R -$ *is* exact. If $M = RS$, the free R-module on a set S, then $M \otimes_R N = \oplus_S N$, since tensoring distributes over direct sums. Then we remember the following "obvious" fact:

Lemma 21.2. *If $M_i' \to M_i \to M_i''$ is exact for all $i \in I$, then so is*

$$\bigoplus M_i' \to \bigoplus M_i \to \bigoplus M_i''\,.$$

Proof. Clearly the composite is zero. Let $(x_i \in M_i, i \in I) \in \bigoplus M_i$ and suppose it maps to zero. That means that each x_i maps to zero in M_i'' and hence is in the image of some $x_i' \in M_i'$. Just make sure to take $x_i' = 0$ if $x_i = 0$. □

To exploit this observation, we'll "resolve" M by free modules. This means: find a surjection from a free R-module, $F_0 \to M$. This amounts to specifying R-module generators. For a general ring R, the kernel of $F_0 \to M$ may not be free. For the moment, let's make sure that it is by assuming that R is a PID, and write F_1 for the kernel. The failure of $M \otimes -$ to be exact is measured, at least partially, by the leftmost term (defined as a kernel) in the exact sequence

$$0 \to \mathrm{Tor}_1^R(M, N) \to F_1 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0\,.$$

The notation suggests that this Tor term is independent of the resolution. This is indeed the case, as we shall show presently. But before we do, let's compute some Tor groups.

Example 21.3. For any PID R, if $M = F$ is free over R we can take $F_0 = F$ and $F_1 = 0$, and discover that then $\mathrm{Tor}_1^R(F, N) = 0$ for any N.

Example 21.4. Let $R = \mathbf{Z}$ and $M = \mathbf{Z}/n\mathbf{Z}$, and N any abelian group. When $R = \mathbf{Z}$ it is often omitted from the notation for Tor. There is a nice free resolution staring at us: $F_0 = F_1 = \mathbf{Z}$, and $F_1 \to F_0$ given by multiplication by n. The sequence defining Tor_1 looks like

$$0 \to \mathrm{Tor}_1(\mathbf{Z}/n\mathbf{Z}, N) \to \mathbf{Z} \otimes N \xrightarrow{n\otimes 1} \mathbf{Z} \otimes N \to \mathbf{Z}/n\mathbf{Z} \otimes N \to 0\,,$$

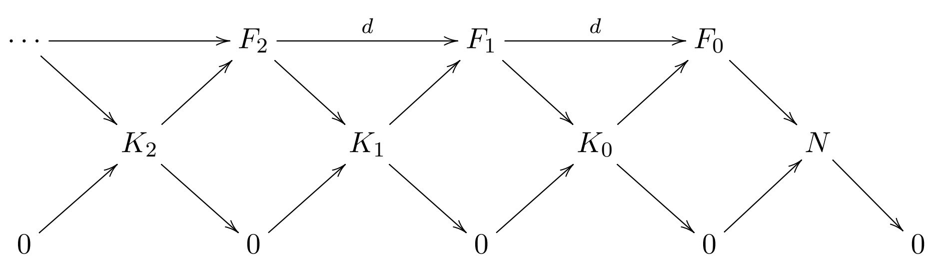
so

$$\mathbf{Z}/n\mathbf{Z} \otimes N = N/nN\,, \quad \mathrm{Tor}_1(\mathbf{Z}/n\mathbf{Z}, N) = \ker(n|N)\,.$$

The torsion in this case is the "n-torsion" in N. This accounts for the name.

Functors like Tor_1 can be usefully defined for any ring, and moving to that general case makes their significance clearer and illuminates the reason why Tor_1 is independent of choice of generators.

So let R be any ring and M a module over it. By picking R-module generators I can produce a surjection from a free R-module, $F_0 \to M$. Write K_0 for the kernel of this map. It is the module of relations among the generators. We can no longer guarantee that it's free, but we can at least find a set of module generators for it, and construct a surjection from a free R-module, $F_1 \to K_0$. Continuing in this way, we get a diagram like this –



– in which the upside-down V subdiagrams are short exact sequences and F_s is free for all s. Splicing these exact sequences gives you an exact sequence in the top row. This is a *free resolution of* N. The top row, F_*, is a chain complex. It maps to the very short chain complex with N in degree 0 and 0 elsewhere, and this chain map is a homology isomorphism (or "quasi-isomorphism"). We have in effect replaced N with this chain complex of free modules. The module N may be very complicated, with generators, relations, relations between relations .... All this is laid out in front of us by the free resolution. Generators of F_0 map to generators for N, and generators for F_1 map to relations among those generators.

Now we can try to define higher Tor functors by tensoring F_* with N and taking homology. If R is a PID and the resolution is just $F_1 \to F_0$, forming homology is precisely taking cokernel and kernel, as we did above. In general, we define

$$\mathrm{Tor}_n^R(M, N) = H_n(M \otimes_R F_*) \,.$$

In the next lecture we will check that this is well-defined – independent of free resolution, and functorial in the arguments. For the moment, notice that

$$\mathrm{Tor}_n^R(M, F) = 0 \quad \text{for}\, n > 0 \quad \text{if } F \text{ is free}\,,$$

since I can take $F \xleftarrow{\cong} F \leftarrow 0 \leftarrow \cdots$ as a free resolution; and that

$$\mathrm{Tor}_0^R(M, N) = M \otimes_R N$$

since we know that $M \otimes_R -$ is right-exact.