22 The fundamental theorem of homological algebra

We will now show that the R-modules $\mathrm{Tor}_n^R(M,N)$ are well-defined and functorial. This will be an application of a very general principle.

Theorem 22.1 (Fundamental Theorem of Homological Algebra)**.** *Let M and N be R-modules; let*

$$0 \leftarrow M \leftarrow E_0 \leftarrow E_1 \leftarrow \cdots$$

be a sequence in which each E_n is free; let

$$0 \leftarrow N \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$$

be an exact sequence; and let $f : M \to N$ be a homomorphism. Then we can lift f to a chain map $f_* : E_* \to F_*$, uniquely up to chain homotopy.

Proof. Let's try to construct f_0. Consider:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0 = \ker(\epsilon_M) & \longrightarrow & E_0 & \xrightarrow{\epsilon_M} & M & & \\
 & & \downarrow g_0 & & \downarrow f_0 & & \downarrow f & & \\
0 & \longrightarrow & L_0 = \ker(\epsilon_N) & \longrightarrow & F_0 & \xrightarrow{\epsilon_N} & N & \longrightarrow & 0
\end{array}$$

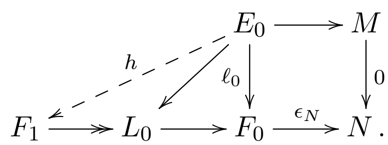
We know that $E_0 = RS$ for some set S. What we do is map the generators of E_0 into M via ϵ_M and then into F via f, and then lift them to F_0 via ϵ_N (which is possible because it's surjective). Then extend to a homomorphism, to get f_0. You can restrict f_0 to kernels to get g_0.

Now the map $d : E_1 \to E_0$ satisifes $\epsilon_M \circ d = 0$, and so factors through a map to $K_0 = \ker \epsilon_M$. Similarly, $d : F_1 \to F_0$ factors through a map $F_1 \to L_0$, and this map must be surjective because the sequence $F_1 \to F_0 \to N$ is exact. We find ourselves in exactly the same situation:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & E_1 & \longrightarrow & K_0 & & \\
 & & \downarrow g_1 & & \downarrow f_1 & & \downarrow g_0 & & \\
0 & \longrightarrow & L_1 & \longrightarrow & F_1 & \longrightarrow & L_0 & \longrightarrow & 0
\end{array}$$

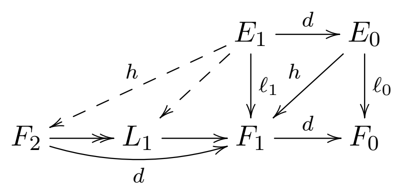
So we construct f_* by induction.

Now we need to prove the chain homotopy claim. So suppose I have $f_*, f'_* : E_* \to F_*$, both lifting $f : M \to N$. Then $f'_n - f_n$ (which we'll rename ℓ_n) is a chain map lifting $0 : M \to N$. We want to consruct a chain null-homotopy of ℓ_*; that is, we want $h : E_n \to F_{n+1}$ such that $dh + hd = \ell_n$. At the bottom, $E_{-1} = 0$, so we want $h : E_0 \to F_1$ such that $dh = \ell_0$. This factorization happens in two steps.



First, $\epsilon_N \ell_0 = 0$ implies that ℓ_0 factors through $L_0 = \ker \epsilon_N$. Next, $F_1 \to L_0$ is surjective, by exactness, and E_0 is free, so we can lift generators and extend R-linearly to get $h : E_0 \to F_1$..

The next step is organized by the diagram



This diagram doesn't commute; $dh = \ell_0$, but the (d, h, ℓ_1) triangle doesn't commute. Rather, we want to construct $h : E_1 \to F_2$ such that $dh = \ell_1 - hd$. Since

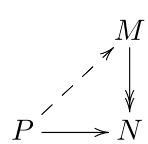
$$d(\ell_1 - hd) = \ell_0 d - dhd = (\ell_0 - dh)d = 0 \,.$$

the map $\ell_1 - hd$ lifts to $L_1 = \ker d$. But then it lifts through F_2, since $F_2 \to L_1$ is surjective and E_1 is free.

Exactly the same process continues. □

This proof uses a property of freeness that is shared by a broader class of modules.

Definition 22.2. An R-module P is *projective* if any map out of P factors through any surjection:



Every free module is projective, and this is the property of freeness that we jave been using; the Fundamental Theorem of Homological Algebra holds under the weaker assumption that each E_n is projective.

Any direct summand in a projective is also projective. Any projective module is a direct summand of a free module. Over a PID, every projective is free, because any submodule of a free is free. But there are examples of nonfree projectives:

Example 22.3. Let k be a field and let R be the product ring $k \times k$. It acts on k in two ways, via $(a, b)c = ac$ and via $(a, b)c = bc$. These are both projective R-modules that are not free.

Now we will apply Theorem 22.1 to verify that our proposed construction of Tor is independent of free (or projective!) resolution, and is functorial.

Suppose I have $f : N' \to N$. Pick arbitrary free resolutions $N' \leftarrow F'_*$ and $N \leftarrow F_*$, and pick any chain map $f_* : F'_* \to F_*$ lifting f. We claim that the map induced in homology by $1 \otimes f_* : M \otimes_R F'_* \to M \otimes_R F_*$ is independent of the choice of lift. Suppose f'_* is another lift, and pick a chain homotopy $h : f_* \simeq f'_*$. Since $M \otimes_R -$ is additive, the relation

$$1 \otimes h : 1 \otimes f_* \simeq 1 \otimes f'_*$$

still holds. So $1 \otimes f_*$ and $1 \otimes f'_*$ induce the same map in homology.

For example, suppose that F_* and F'_* are two projective resolutions of N. Any two lifts of the identity map are chain-homotopic, and so induce the same map $H_*(M \otimes_R F_*) \to H_*(M \otimes_R F'_*)$. So if $f : F_* \to F'_*$ and $g : F'_* \to F_*$ are chain maps lifing the identity, then $f_* \circ g_*$ induces the same self-map of $H_*(M \otimes_R F'_*)$ as the identity self-map does, and so (by functoriality) is the identity. Similarly, $g_* \circ f_*$ induces the identity map on $H_*(M \otimes_R F_*)$. So they induce inverse isomorphisms.

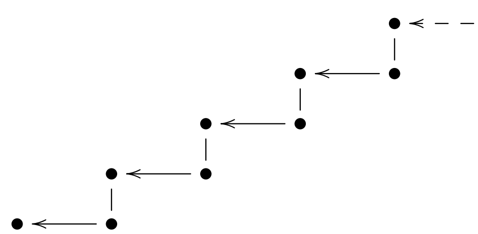
Putting all this together shows that any two projective resolutions of N induce canonically isomorphic modules $\mathrm{Tor}_n^R(M, N)$, and that a homomorphism $f : N' \to N$ induces a well defined map $\mathrm{Tor}_n^R(M, N') \to \mathrm{Tor}_n^R(M, N)$ that renders $\mathrm{Tor}_n^R(M, -)$ a functor.

My last comment about Tor is that there's a symmetry there. Of course, $M \otimes_R N \cong N \otimes_R M$. This uses the fact that R is commutative. This leads right on to saying that $\mathrm{Tor}_n^R(M, N) \cong \mathrm{Tor}_n^R(N, M)$. We've been computing Tor by taking a resolution of the second variable. But I could equally have taken a resolution of the first variable. This follows from Theorem 22.1.

Example 22.4. I want to give an example when you do have higher Tor modules. Let k be a field, and let $R = k[d]/(d^2)$. This is sometimes called the "dual numbers," or the exterior algebra over k. What is an R-module? It's just a k-vector space M with an operator d (given by multiplication by d) that satisfies $d^2 = 0$. Even though there's no grading around, I can still define the "homology" of M:

$$H(M; d) = \frac{\ker d}{\operatorname{im} d}\,.$$

This k-algebra is *augmented* by an algebra map $\epsilon : R \to k$ splitting the unit; $\epsilon(d) = 0$. This renders k an R-module. Let's construct a free R-module resolution of this module. Here's a picture.



The vertical lines indicate multiplication by d. We could write this as

$$0 \leftarrow k \overset{\epsilon}{\leftarrow} R \overset{d}{\leftarrow} R \overset{d}{\leftarrow} R \leftarrow \cdots\,.$$

Now tensor this over R with an R-module M; so M is a vector space equipped with an operator d with $d^2 = 0$. Each copy of R gets replaced by a copy of M, and the differential gives multiplication by d on M. So taking homology gives

$$\mathrm{Tor}_n^R(k, M) = \begin{cases} k \otimes_R M = M/dM & \text{for } n = 0 \\ H(M; d) & \text{for } n > 0\,. \end{cases}$$

So for example

$$\mathrm{Tor}_n^R(k, k) = k \quad \text{for } n \geq 0\,.$$