24 Universal coefficient theorem

Suppose that we are given $H_*(X;\mathbf{Z})$. Can we compute $H_*(X;\mathbf{Z}/2\mathbf{Z})$? This is non-obvious. Consider the map $\mathbf{RP}^2 \to S^2$ that pinches $\mathbf{RP}^1$ to a point. Now $H_2(\mathbf{RP}^2;\mathbf{Z}) = 0$, so in H_2 this map is zero. But in $\mathbf{Z}/2\mathbf{Z}$-coefficients, in dimension 2, this map gives an isomorphism. This shows that there's no *functorial* determination of $H_*(X;\mathbf{Z}/2)$ in terms of $H_*(X;\mathbf{Z})$; the effect of a map in integral homology does not determine its effect in mod 2 homology. So how *do* we go between different coefficients?

Let R be a commutative ring and M an R-module, and suppose we have a chain complex C_* of R-modules. It could be the singular complex of a space, but it doesn't have to be. Let's compare $H_n(C_*) \otimes M$ with $H_n(C_* \otimes M)$. (Here and below we'll just write $\otimes$ for $\otimes_R$.) The latter thing gives homology with coefficients in M. How can we compare these two? Let's investigate, and build up conditions on R and C_* as we go along.

First, there's a natural map

$$\alpha : H_n(C_*) \otimes M \to H_n(C_* \otimes M)\,,$$

sending $[z] \otimes m$ to $[z \otimes m]$. We propose to find conditions under which it is injective. The map α fits into a commutative diagram with exact columns like this:

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H_n(C_*) \otimes M & \xrightarrow{\alpha} & H_n(C_* \otimes M) \\
\uparrow & & \uparrow \\
Z_n(C_*) \otimes M & \longrightarrow & Z_n(C_* \otimes M) \\
\uparrow & & \uparrow \\
C_{n+1} \otimes M & \xrightarrow{=} & C_{n+1} \otimes M\,.
\end{array}$$

Now, $Z_n(C_* \otimes M)$ is a submodule of $C_n \otimes M$, but the map $Z_n(C) \otimes M \to C_n \otimes M$ need not be injective . . . unless we impose more restrictions. If we can guarantee that it is, then a diagram chase shows that α is a monomorphism.

So let's assume that R is a PID and that C_n is a free R-module for all n. Then the submodule $B_{n-1}(C_*) \subseteq C_{n-1}$ is again free, so the short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_n(C_*) & \longrightarrow & C_n & \longrightarrow & B_{n-1}(C_*) & \longrightarrow & 0 \\
 & & & & & \searrow^{d} & \downarrow & & \\
 & & & & & & C_{n-1} & &
\end{array}$$

splits. So $Z_n(C_*) \to C_n$ is a split monomorphism, and hence $Z_n(C_*) \otimes M \to C_n \otimes M$ is too.

In fact, a little thought shows that this argument produces a splitting of the map α.

Now, α is not always an isomorphism. But it certainly is if $M = R$, and it's compatible with direct sums, so it certainly is if M is free. The idea is now to resolve M by frees, and see where that idea takes us.

So let

$$0 \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M. Again, we're using the assumption that R is a PID, to guarantee that $\ker(F_0 \to M)$ is free. Again using the assumption that each C_n is free, we get a short exact sequence of chain complexes

$$0 \to C_* \otimes F_1 \to C_* \otimes F_0 \to C_* \otimes M \to 0\,.$$

In homology, this gives a long exact sequence. Unsplicing it gives the left-hand column in the following diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\operatorname{coker}(H_n(C_* \otimes F_1) \to H_n(C_* \otimes F_0)) & \xrightarrow{\cong} & \operatorname{coker}(H_n(C_*) \otimes F_1 \to H_n(C_*) \otimes F_0)) \\
\downarrow & & \downarrow \\
H_n(C_* \otimes M) & \xrightarrow{=} & H_n(C_* \otimes M) \\
\downarrow{\scriptstyle \partial} & & \downarrow \\
\ker(H_{n-1}(C_* \otimes F_1) \to H_{n-1}(C_* \otimes F_0)) & \xrightarrow{\cong} & \ker(H_{n-1}(C_*) \otimes F_1 \to H_{n-1}(C_*) \otimes F_0) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The right hand column occurs because α is an isomorphism when the module involved is free. But

$$\operatorname{coker}(H_n(C_*) \otimes F_1 \to H_n(C_*) \otimes F_0)) = H_n(C_*) \otimes M$$

and

$$\ker(H_{n-1}(C_*) \otimes F_1 \to H_{n-1}(C_*) \otimes F_0) = \operatorname{Tor}_1^R(H_{n-1}(C_*), M)\,.$$

We have proved the following theorem.

Theorem 24.1 (Universal Coefficient Theorem)**.** *Let R be a PID and C_* a chain complex of R-modules such that C_n is free for all n. Then there is a natural short exact sequence of R-modules*

$$0 \to H_n(C_*) \otimes M \xrightarrow{\alpha} H_n(C_* \otimes M) \xrightarrow{\partial} \operatorname{Tor}_1^R(H_{n-1}(C_*), M) \to 0$$

that splits (but not naturally).

Example 24.2. The pinch map $\mathbf{RP}^2 \to S^2$ induces the following map of universal coefficient short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\mathbf{RP}^2) \otimes \mathbf{Z}/2\mathbf{Z} & \longrightarrow & H_2(\mathbf{RP}^2; \mathbf{Z}/2\mathbf{Z}) & \xrightarrow{\cong} & \operatorname{Tor}_1(H_1(\mathbf{RP}^2), \mathbf{Z}/2\mathbf{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & H_2(S^2) \otimes \mathbf{Z}/2\mathbf{Z} & \xrightarrow{\cong} & H_2(S^2; \mathbf{Z}/2\mathbf{Z}) & \longrightarrow & \operatorname{Tor}_1(H_1(S^2), \mathbf{Z}/2\mathbf{Z}) & \longrightarrow & 0
\end{array}$$

This shows that the splitting of the universal coefficient short exact sequence cannot be made natural, and it explains the mystery that we began with.

Exercise 24.3. The hypotheses are essential. Construct two counterexamples: one with $R = \mathbf{Z}$ but in which the groups in the chain complex are not free, and one in which $R = k[d]/d^2$ and the modules in C_* are free over R.