Chapter 3

Cohomology and duality

26 Coproducts, cohomology

The next topic is cohomology. This is like homology, but it's a contravariant rather than covariant functor of spaces. There are three reasons why you might like a contravariant functor.
(1) Many geometric contructions *pull back*; that is, they behave contravariantly. For example, if I have some covering space $\widetilde{X} \to X$ and a map $f : Y \to X$, I get a pullback covering space $f^*\widetilde{X}$. A better example is vector bundles (that we'll talk about in 18.906) – they don't push out, they pullback. So if we want to study them by means of "natural" invariants, these invariants will have to lie in a (hopefully computable) group that also behaves contravariantly. This will lead to the theory of *characteristic classes*.
(2) The structure induced by the diagonal map from a space to its square induces stucture in contravariant functors that is more general and easier to study.
(3) Cohomology turns out to be the target of the Poincaré duality map.

Let's elaborate on point (2). Every space has a diagonal map

$$X \xrightarrow{\Delta} X \times X \,.$$

This induces a map $H_*(X; R) \to H_*(X \times X; R)$, for any coefficient group R. Now, if R is a ring, we get a cross product map

$$\times : H_*(X; R) \otimes_R H_*(X; R) \to H_*(X \times X; R) \,.$$

If R is a PID, the Künneth Theorem tells us that this map is a monomorphism. If the remaining term in the Künneth Theorem is zero, the cross product is an isomorphism. So if $H_*(X; R)$ is free over R (or even just flat over R), we get a "diagonal" or "coproduct"

$$\Delta : H_*(X; R) \to H_*(X; R) \otimes_R H_*(X; R) \,.$$

If R is a field, this map is universally defined, and natural in X.

This kind of structure is unfamiliar, and at first seems a bit strange. After all, the tensor product is defined by a universal property for maps *out* of it; maps *into* it just are what they are.

Still, it's often useful, and we pause to fill in some of its properties.

Definition 26.1. Let R be a ring. A *(graded) coalgebra* over R is a (graded) R-module M equipped with a "comultiplication" $\Delta : M \to M \otimes_R M$ and a "counit" map $\varepsilon : M \to R$ such that the following diagrams commute.

$$\begin{array}{ccccc} & & M & & \\ & \overset{=}{\swarrow} & \downarrow{\scriptstyle\Delta} & \overset{=}{\searrow} & \\ R\otimes_R M & \xleftarrow{\varepsilon\otimes 1} & M\otimes_R M & \xrightarrow{1\otimes\varepsilon} & M\otimes_R R \end{array}$$

$$\begin{array}{ccc} M & \xrightarrow{\Delta} & M\otimes_R M \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\otimes 1} \\ M\otimes_R M & \xrightarrow{1\otimes\Delta} & M\otimes_R M\otimes_R M \end{array}$$

It is *commutative* if in addition

$$\begin{array}{ccccc} & & M & & \\ & \overset{\Delta}{\swarrow} & & \overset{\Delta}{\searrow} & \\ M\otimes_R M & & \xrightarrow{\tau} & & M\otimes_R M \end{array}$$

commutes, where $\tau(x\otimes y)=(-1)^{|x|\cdot|y|}y\otimes x$ is the twist map.

Using acyclic models, you saw for homework that the the Künneth map is associative and commutative: The diagrams

$$\begin{array}{ccc} S_*(X)\otimes S_*(Y)\otimes S_*(Z) & \xrightarrow{\times\otimes 1} & S_*(X\times Y)\otimes S_*(Z) \\ \downarrow{\scriptstyle 1\otimes\times} & & \downarrow{\scriptstyle\times} \\ S_*(X)\otimes S_*(Y\times Z) & \xrightarrow{\times} & S_*(X\times Y\times Z) \end{array}$$

and

$$\begin{array}{ccc} S_*(X)\otimes S_*(Y) & \xrightarrow{\tau} & S_*(Y)\otimes S_*(X) \\ \downarrow{\scriptstyle\times} & & \downarrow{\scriptstyle\times} \\ S_*(X\times Y) & \xrightarrow{T_*} & S_*(Y\times X) \end{array}$$

commute up to natural chain homotopy, where τ is as defined above on the tensor product and $T: X\times Y\to Y\times X$ is the swap map. Similar diagrams apply to the standard comparison map for the homology of tensor products of chain complexes,

$$\mu: H_*(C)\otimes H_*(D)\to H_*(C\otimes D)\,,$$

and the result is this:

Corollary 26.2. *Suppose R is a PID and $H_*(X;R)$ is free over R. Then $H_*(X;R)$ has the natural structure of a commutative graded coalgebra over R.*

We could now just go on and talk about coalgebras. But they are less familiar, and available only if $H_*(X;R)$ is free over R. So instead we're going to dualize, talk about cohomology, and get an algebra structure. Some say that cohomology is better because you have algebras, but that's more of a sociological statement than a mathematical one.

Let's get on with it.

Definition 26.3. Let N be an abelian group. A *singular n-cochain* on X with values in N is a function $\mathrm{Sin}_n(X) \to N$.

If N is an R-module, then I can extend linearly to get an R-module homomorphism $S_n(X;R) \to N$.

Notation 26.4. Write

$$S^n(X;N) = \mathrm{Map}(\mathrm{Sin}_n(X), N) = \mathrm{Hom}_R(S_n(X;R), N)\,.$$

This is going to give us something contravariant, that's for sure. But we haven't quite finished dualizing. The differential $d : S_{n+1}(X;R) \to S_n(X;R)$ induces a "coboundary map"

$$d : S^n(X;N) \to S^{n+1}(X;N)$$

defined by

$$(df)(\sigma) = (-1)^{n+1} f(d\sigma)\,.$$

The sign is a little strange, and we'll see an explanation in a minute. Anyway, we get a "cochain complex," with a differential that *increases* degree by 1. We still have $d^2 = 0$, since

$$(d^2 f)(\sigma) = \pm d(f(d\sigma)) = \pm f(d^2\sigma) = \pm f(0) = 0\,,$$

so we can still take homology of this cochain complex.

Definition 26.5. The nth *singular cohomology group* of X with coefficients in an abelian group N is

$$H^n(X;N) = \frac{\ker(S^n(X;N) \to S^{n+1}(X;N))}{\mathrm{im}(S^{n-1}(X;N) \to S^n(X;N))}\,.$$

If N is an R-module, then $H^n(X;N)$ is again an R-module.

Let's first compute $H^0(X;N)$. A 0-cochain is a function $\mathrm{Sin}_0(X) \to N$; that is, a function (not required to be continuous!) $f : X \to N$. To compute df, take a 1-simplex $\sigma : \Delta^1 \to X$ and evaluate f on its boundary:

$$(df)(\sigma) = -f(d\sigma) = -f(\sigma(e_0) - \sigma(e_1)) = f(\sigma(e_1)) - f(\sigma(e_0))\,.$$

So f is a co*cycle* if it's constant on path components. That is to say:

Lemma 26.6. $H^0(X;N) = \mathrm{Map}(\pi_0(X), N)$.

Warning 26.7. $S^n(X;\mathbf{Z}) = \mathrm{Map}(\mathrm{Sin}_n(X);\mathbf{Z}) = \prod_{\mathrm{Sin}_n(X)} \mathbf{Z}$, which is probably an uncountable product. An awkward fact is that this is never free abelian.

The first thing a cohomology class does is to give a linear functional on homology, by "evaluation." Let's spin this out a bit.

We want to tensor together cochains and chains. But to do that we should make the differential in $S^*(X)$ go down, not up. Just as a notational matter, let's write

$$S^\vee_{-n}(X;N) = S^n(X;N)$$

and define a differential $d : S^\vee_{-n}(X) \to S^\vee_{-n-1}(X)$ to be the differential $d : S^n(X) \to S^{n+1}(X)$. Now $S^\vee_*(X)$ is a chain complex, albeit a negatively graded one. Form the graded tensor product, with

$$\left(S^\vee_*(X;N) \otimes S_*(X)\right)_n = \bigoplus_{p+q=n} S^\vee_p(X;N) \otimes S_q(X)\,.$$

Now evaluation is a map of graded abelian groups

$$\langle -, - \rangle : S_*^\vee(X; N) \otimes S_*(X) \to N \,,$$

where N is regarded as a chain complex concentrated in degree 0. We would like this map to be a chain map. So let $f \in S^n(X; N)$ and $\sigma \in S_n(X)$, and compute

$$0 = d\langle f, \sigma \rangle = \langle df, \sigma \rangle + (-1)^n \langle f, d\sigma \rangle \,.$$

This forces

$$(df)(\sigma) = \langle df, \sigma \rangle = -(-1)^n f(d\sigma) \,,$$

explaining the odd sign in our definition above.

Here's the payoff: There's a natural map

$$H_{-n}(S_*^\vee(X; N)) \otimes H_n(S_*(X)) \xrightarrow{\mu} H_0\left(S_*^\vee(X; N) \otimes S_*(X)\right) \to N$$

This gives us the *Kronecker pairing*

$$\langle -, - \rangle : H^n(X; N) \otimes H_n(X) \to N \,.$$

We can develop the properties of cohomology in analogy with properties of homology. For example: If $A \subseteq X$, there is a restriction map $S^n(X; N) \to S^n(A; N)$, induced by the injection $\mathrm{Sin}_n(A) \hookrightarrow \mathrm{Sin}_n(X)$. And as long as A is nonempty, we can split this injection, so any function $\mathrm{Sin}_n(A) \to N$ extends to $\mathrm{Sin}_n(X) \to N$. This means that $S^n(X; N) \to S^n(A; N)$ is surjective. (This is the case if $A = \varnothing$, as well!)

Definition 26.8. The *relative n-cochain group* with coefficients in N is

$$S^n(X, A; N) = \ker\left(S^n(X; N) \to S^n(A; N)\right) \,.$$

This defines a sub cochain complex of $S^*(X; N)$, and we define

$$H^n(X, A; N) = H^n(S^*(X, A; N)) \,.$$

The short exact sequence of cochain complexes

$$0 \to S^*(X, A; N) \to S^*(X; N) \to S^*(A; N) \to 0$$

induces the *long exact cohomology sequence*

$$\begin{array}{ccccc}
\cdots & \xleftarrow{\quad\delta\quad} & & & \\
H^1(X, A; N) & \longrightarrow & H^1(X; N) & \longrightarrow & H^1(A; N) \\
& \xleftarrow{\quad\delta\quad} & & & \\
H^0(X, A; N) & \longrightarrow & H^0(X; N) & \longrightarrow & H^0(A; N) \,.
\end{array}$$

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