27 Ext and UCT

Let R be a ring (probably a PID) and N an R-module. The singular cochains on X with values in N,

$$S^*(X;N) = \mathrm{Map}(\mathrm{Sin}_*(X), N)\,,$$

then forms a cochain complex of R-modules. It is contravariantly functorial in X and covariantly functorial in N. The Kronecker pairing defines a map

$$H^n(X;N) \otimes_R H_n(X;R) \to N$$

whose adjoint

$$\beta : H^n(X;N) \to \mathrm{Hom}_R(H_n(X;R), N)$$

gives us an estimate of the cohomology in terms of the homology of X. Here's how well it does:

Theorem 27.1 (Mixed variance Universal Coefficient Theorem)**.** *Let R be a PID and N an R-module, and let C_* be a chain-complex of free R-modules. Then there is a short exact sequence of R-modules,*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*), N) \to H^n(\mathrm{Hom}_R(C_*, N)) \to \mathrm{Hom}_R(H_n(C_*), N) \to 0\,,$$

natural in C_* and N, that splits (but not naturally).

Taking $C_* = S_*(X;R)$, we have the short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R), N) \to H^n(X;N) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(X;R), N) \to 0$$

that splits, but not naturally. This also holds for relative cohomology.

What is this Ext?

The problem that arises is that $\mathrm{Hom}_R(-, N) : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is not exact. Suppose I have an injection $M' \to M$. Is $\mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N)$ surjective? Does a map $M' \to N$ necessarily extend to a map $M \to N$? No! For example, $\mathbf{Z}/2\mathbf{Z} \hookrightarrow \mathbf{Z}/4\mathbf{Z}$ is an injection, but the identity map $\mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ does not extend over $\mathbf{Z}/4\mathbf{Z}$.

On the other hand, if $M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ is an exact sequence of R-modules then

$$0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$$

is again exact. Check this statement!

Now homological algebra comes to the rescue to repair the failure of exactness. Pick a free resolution of M,

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_2 \leftarrow \cdots\,.$$

Apply $\mathrm{Hom}(-, N)$ to get a cochain complex

$$0 \to \mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N) \to \mathrm{Hom}_R(F_2, N) \to \cdots\,.$$

Definition 27.2. $\mathrm{Ext}^n_R(M,N) = H^n(\mathrm{Hom}_R(F_*, N))$.

Remark 27.3. Ext is well-defined and functorial, by the Fundamental Theorem of Homological Algebra, Theorem 22.1. If M is free (or projective) then $\mathrm{Ext}^n_R(M,-) = 0$ for $n > 0$, since we can take M as its own projective resolution. If R is a PID, then we can assume $F_1 = \ker(F_0 \to M)$ and $F_n = 0$ for $n > 1$, so $\mathrm{Ext}^n_R = 0$ if $n > 1$. If R is a field, then $\mathrm{Ext}^n_R = 0$ for $n > 0$.

Example 27.4. Let $R = \mathbf{Z}$ and take $M = \mathbf{Z}/k\mathbf{Z}$. This admits a simple free resolution: $0 \to \mathbf{Z} \xrightarrow{k} \mathbf{Z} \to \mathbf{Z}/k\mathbf{Z} \to 0$. Apply $\mathrm{Hom}(-, N)$ to it, and remember that $\mathrm{Hom}(\mathbf{Z}, N) = N$, to get the very short cochain complex, with entries in dimensions 0 and 1:

$$0 \to N \xrightarrow{k} N \to 0 \,.$$

Taking homology gives us

$$\mathrm{Hom}(\mathbf{Z}/k\mathbf{Z}, N) = \ker(k|N) \qquad \mathrm{Ext}^1(\mathbf{Z}/k\mathbf{Z}, N) = N/kN \,.$$

Proof of Theorem 27.1. First of all, we can't just copy the proof (in Lecture 24) of the homology universal coefficient theorem, since $\mathrm{Ext}^1_R(-, R)$ is not generally trivial.

Instead, we start by thinking about what an n-cocycle in $\mathrm{Hom}_R(C_*, N)$ is: it's a homomorphism $C_n \to N$ such that the composite $C_{n+1} \to C_n \to N$ is trivial. Write $B_n \subseteq C_n$ for the submodule of boundaries. We have a homomorphism that kills B_n; that is,

$$Z^n(\mathrm{Hom}_R(C_*, N)) \xrightarrow{\cong} \mathrm{Hom}_R(C_n/B_n, N) \,.$$

Now $H_n(C_*)$ (which we'll abbreviate as H_n) is the submodule Z_n/B_n of C_n/B_n; we have an exact sequence

$$0 \to H_n \to C_n/B_n \to B_{n-1} \to 0 \,.$$

Apply $\mathrm{Hom}_R(-, N)$ to this short exact sequence. The result is again short exact, because B_{n-1} is a submodule of the free R-module C_{n-1} and hence is free. This gives us the bottom line in the map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^n \mathrm{Hom}_R(C_*, N) & \longrightarrow & Z^n \mathrm{Hom}_R(C_*, N) & \longrightarrow & H^n(\mathrm{Hom}_R(C_*, N)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow \beta & & \\
0 & \longrightarrow & \mathrm{Hom}_R(B_{n-1}, N) & \longrightarrow & \mathrm{Hom}_R(C_n/B_n, N) & \longrightarrow & \mathrm{Hom}_R(H_n, N) & \longrightarrow & 0 \,.
\end{array}$$

The map β is the one we started with. The snake lemma now shows that it is surjective and that

$$\ker \beta \cong \mathrm{coker}(B^n \mathrm{Hom}_R(C_*, N) \to \mathrm{Hom}_R(B_{n-1}, N)) \,.$$

An element of $B^n \mathrm{Hom}_R(C_*, N)$ is a map $C_n \to N$ that factors as $C_n \xrightarrow{d} C_{n-1} \to N$. The observation is now that this is the same as factoring as $C_n \xrightarrow{d} Z_{n-1} \to N$; once this factorization has been achieved, the map $Z_{n-1} \to N$ automatically extends to all of C_{n-1}. This is because $Z_{n-1} \subseteq C_{n-1}$ as a direct summand: the short exact sequence

$$0 \to Z_{n-1} \to C_{n-1} \to B_{n-2} \to 0$$

splits since B_{n-2} is free. Consequently we can rewrite our forumula for $\ker \beta$ as

$$\ker \beta \cong \mathrm{coker}(\mathrm{Hom}_R(Z_{n-1}, N) \to \mathrm{Hom}_R(B_{n-1}, N)) \,.$$

But after all

$$0 \leftarrow H_{n-1} \leftarrow Z_{n-1} \leftarrow B_{n-1} \leftarrow 0$$

is a free resolution, so this cokernel is precisely $\mathrm{Ext}^1_R(H_{n-1}(C_*), N)$. □

Question 27.5. Why is Ext called Ext?
Answer: It classifies extensions. Let R be a commutative ring, and let M, N be two R-modules. I can think about "extensions of M by N," that is, short exact sequences of the form

$$0 \to N \to L \to M \to 0\,.$$

For example, I have two extensions of $\mathbf{Z}/2\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$:

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

and

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0\,.$$

We'll say that two extensions are *equivalent* if there's a map of short exact sequences between them that is the identity on N and on M. The two extensions above aren't equivalent, for example.

Another definition of $\mathrm{Ext}^1_R(M,N)$ is: the set of extensions like this modulo this notion of equivalence. The zero in the group is the split extension.

The universal coefficient theorem is useful in transferring properties of homology to cohomology. For example, if $f : X \to Y$ is a map that induces an isomorphism in $H_*(-;R)$, then it induces an isomorphism in $H^*(-;N)$ for any R-module N, at least provided that R is a PID. (This is in fact true in general.)

Cohomology satisfies the appropriate analogues of the Eilenberg-Steenrod axioms.
Homotopy invariance: If $f_0 \simeq f_1 : (X,A) \to (Y,B)$, then

$$f_0^* = f_1^* : H^*(Y,B;N) \to H^*(X,A;N)\,.$$

I can't use the UCT to address this. But we established a chain homotopy $f_{0,*} \simeq f_{1,*} : S_*(X,A) \to S_*(Y,B)$, and applying Hom converts chain homotopies to cochain homotopies.
Excision: If $U \subseteq A \subseteq X$ such that $\overline{U} \subseteq \mathrm{Int}(A)$, then $H^*(X,A;N) \to H^*(X-U, A-U;N)$ is an isomorphism. This follows from excision in homology and the mixed variance UCT.
Milnor axiom: The inclusions induce an isomorphism

$$H^*(\coprod_\alpha X_\alpha;N) \to \prod_\alpha H^*(X_\alpha;N)\,.$$

As a result, it enjoys the fruit of these axioms, such as:
The Mayer-Vietoris sequence: If $A, B \subseteq X$ are such that their interiors cover X, then there is a long exact sequence

$$\cdots \to H^{n-1}(A\cap B;N) \to H^n(X;N) \to H^n(A;N)\oplus H^n(B;N) \to H^n(A\cap B;N) \to H^{n+1}(X;N) \to \cdots$$

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