28 Products in cohomology

We'll talk about the cohomology cross product first. The first step is to produce a map on chains that goes in the reverse direction from the cross product we constructed in Lecture 7.

Construction 28.1. For each pair of natural numbers p, q, we will define a natural homomorphism

$$\alpha : S_{p+q}(X \times Y) \to S_p(X) \otimes S_q(Y) .$$

It suffices to define this on simplices, so let $\sigma : \Delta^{p+q} \to X \times Y$ be a singular $(p+q)$-simplex in the product. Let

$$\sigma_1 = \mathrm{pr}_1 \circ \sigma : \Delta^{p+q} \to X \quad \text{and} \quad \sigma_2 = \mathrm{pr}_2 \circ \sigma : \Delta^{p+q} \to Y$$

be the two coordinates of σ. I have to produce a p-simplex in X and a q-simplex in Y.

First define two maps in the simplex category:

– the "front face" $\alpha_p : [p] \to [p+q]$, sending i to i for $0 \le i \le p$, and

– the "back face" $\omega_q : [q] \to [p+q]$, sending j to $j+p$ for $0 \le j \le q$.

Use the same symbols for the affine extensions to maps $\Delta^p \to \Delta^{p+q}$ and $\Delta^q \to \Delta^{p+q}$. Now let

$$\alpha(\sigma) = (\sigma_1 \circ \alpha_p) \otimes (\sigma_2 \circ \omega_q) .$$

This seems like a very random construction; but it works! It's named after two great early algebraic topologists, James W. Alexander and Hassler Whitney. For homework, you will show that these maps assemble into a chain map

$$\alpha : S_*(X \times Y) \to S_*(X) \otimes S_*(Y) .$$

This works over any ring R. To get a map in cohomology, we should form a composite

$$S^p(X;R)\otimes_R S^q(Y;R) \to \mathrm{Hom}_R(S_p(X;R)\otimes_R S_q(Y;R), R) \xrightarrow{\alpha^*} \mathrm{Hom}_R(S_{p+q}(X\times Y;R), R) = S^{p+q}(X\times Y;R) .$$

The first map goes like this: Given chain complexes C_* and D_*, we can consider the dual cochain complexes $\mathrm{Hom}_R(C_*, R)$ and $\mathrm{Hom}_R(D_*, R)$, and construct a chain map

$$\mathrm{Hom}_R(C_*, R) \otimes_R \mathrm{Hom}_R(D_*, R) \to \mathrm{Hom}_R(C_* \otimes_R D_*, R)$$

by

$$f \otimes g \mapsto \begin{cases} (x \otimes y \mapsto (-1)^{pq} f(x) g(y)) & |x| = |f| = p,\ |y| = |g| = q \\ 0 & \text{otherwise.} \end{cases}$$

Again, I leave it to you to check that this is a cochain map.

Altogether, we have constructed a natural cochain map

$$\times : S^p(X) \otimes S^q(Y) \to S^{p+q}(X \times Y)$$

From this, we get a homomorphism

$$H^*(S^*(X) \otimes S^*(Y)) \to H^*(X \times Y) .$$

I'm not quite done! As in the Künneth theorem, there is an evident natural map

$$\mu : H^*(X) \otimes H^*(Y) \to H^*(S^*(X) \otimes S^*(Y)) .$$

The composite

$$\times : H^*(X) \otimes H^*(Y) \to H^*(S^*(X) \otimes S^*(Y)) \to H^*(X \times Y)$$

is the *cohomology cross product.*

It's not very easy to do computations with this, directly. We'll find indirect means. Let me make some points about this construction, though.

Definition 28.2. The *cup product* is the map obtained by taking $X = Y$ and composing with the map induced by the diagonal $\Delta : X \to X \times X$:

$$\cup : H^p(X) \otimes H^q(X) \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q}(X), .$$

These definitions make good sense with any ring for coefficients.

Let's explore this definition in dimension zero. I claim that $H^0(X;R) \cong \mathrm{Map}(\pi_0(X), R)$ as rings. When $p = q = 0$, both α_0 and ω_0 are the identity maps, so we are just forming the pointwise product of functions.

There's a distinguished element in $H^0(X)$, namely the the function $\pi_0(X) \to R$ that takes on the value 1 on every path component. This is the identity for the cup product. This comes about because when $p = 0$ in our above story, then α_0 is just including the 0-simplex, and ω_q is the identity.

The cross product is also associative, even on the chain level.

Proposition 28.3. *Let $f \in S^p(X)$, $g \in S^q(Y)$, and $h \in S^r(Z)$, and let $\sigma : \Delta^{p+q+r} \to X \times Y \times Z$ be any simplex. Then*

$$((f \times g) \times h)(\sigma) = (f \times (g \times h))(\sigma) \, .$$

Proof. Write σ_{12} for the composite of σ with the projection map $X \times Y \times Z \to X \times Y$, and so on. Then

$$((f \times g) \times h)(\sigma) = (-1)^{(p+q)r}(f \times g)(\sigma_{12} \circ \alpha_{p+q})h(\sigma_3 \circ \omega_r) \, .$$

But

$$(f \times g)(\sigma_{12} \circ \alpha_{p+q}) = (-1)^{pq} f(\sigma_1 \circ \alpha_p) g(\sigma_2 \circ \mu_q) \, ,$$

where μ_q is the "middle face," sending ℓ to $\ell + p$ for $0 \leq \ell \leq q$. In other words,

$$((f \times g) \times h)(\sigma) = (-1)^{pq+qr+rp} f(\sigma_1 \circ \alpha_p) g(\sigma_2 \circ \mu_q) h(\sigma_3 \circ \omega_r) \, .$$

I've used associativity of the ring. You get exactly the same thing when you expand $(f \times (g \times h))(\sigma)$, so the cross product is associative. □

Of course the diagonal map is "associative," too, and we find that the cup product is associative:

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma) \, .$$