32 Proof of the orientation theorem

We are studying the way in which local homological information gives rise to global information, especially on an n-manifold M. The tool was the map

$$j : H_n(M;R) \to \Gamma(M;o_M \otimes R)$$

sending a class c to the section of the orientation local coefficient system given at $x \in M$ by the restriction $j_x(c) \in H_n(M, M - x)$. We asserted that if M is compact then j is an isomorphism and that $H_q(M) = 0$ for $q > n$. The proof will be by induction.

To make the induction go, we will need a refinement of this construction. Let $A \subseteq M$ be a compact subset. A class in $H_n(M, M - A)$ is represented by a cycle whose boundary lies outside of A. It may cover A evenly. We can give meaning to this question as follows. Let $x \in A$. Then $M - A \subseteq M - x$, so we have a map

$$j_{A,x} : H_n(M, M - A) \to H_n(M, M - x)$$

that tests whether the chain covers x. As x ranges over A, these maps together give us a map to the group of sections of o_M over A,

$$j_A : H_n(M, M - A) \to \Gamma(A; o_M)\,.$$

Because $H_n(M, M - A)$ deals with homology classes that "stretch over A," we will write

$$H_n(M, M - A) = H_n(M|A)\,.$$

Theorem 32.1. *Let M be an n-manifold and let A be a compact subset of M. Then $H_q(M|A;R) = 0$ for $q > n$, and the map $j_A : H_n(M|A;R) \to \Gamma(A; o_M \otimes R)$ is an isomorphism.*

Taking $A = M$ (assuming M compact) we find that $H_q(M;R) = 0$ for $q > n$ and

$$j_M : H_n(M;R) \xrightarrow{\cong} \Gamma(M; o_M \otimes R)\,.$$

But the theorem covers much more exotic situations as well; perhaps A is a Cantor set in some Euclidean space, for example.

We follow [2] in proving this, and refer you to that reference for the modifications appropriate for the more general statement when A is assumed merely closed rather than compact.

First we establish two general results.

Proposition 32.2. *Let A and B be closed subspaces of M, and suppose the result holds for A, B, and $A \cap B$. Then it holds for $A \cup B$.*

Proof. The relative Mayer-Vietoris theorem and the hypothesis that $H_{n+1}(M|A\cap B) = 0$ gives us exactness of the top row in the ladder

$$\begin{array}{ccccccc}
0 \longrightarrow & H_n(M|A\cup B) & \longrightarrow & H_n(M|A) \oplus H_n(M|B) & \longrightarrow & H_n(M|A\cap B) \\
& \downarrow{\scriptstyle j_{A\cap B}} & & \downarrow{\scriptstyle j_A \oplus j_B} & & \downarrow{\scriptstyle j_{A\cap B}} \\
0 \longrightarrow & \Gamma(A\cup B; o_M) & \longrightarrow & \Gamma(A;o_M) \oplus \Gamma(B;o_M) & \longrightarrow & \Gamma(A\cap B; o_M)\,.
\end{array}$$

Exactness of the bottom row is clear: A section over $A \cup B$ is precisely a section over A and a section over B that agree on the intersection. So the five-lemma shows that $j_{A\cup B}$ is an isomorphism. Looking further back in the Mayer-Vietoris sequence gives the vanishing of $H_q(M|A)$ for $q > n$. □

Proposition 32.3. *Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence of compact subsets of M, and assume that the theorem holds for each A_n. Then it holds for the intersection $A = \bigcap A_i$.*

The proof of this proposition entails two lemmas, which we'll dispose of first.

Lemma 32.4. *Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence of compact subsets of a space X, with intersection A. Then*

$$\varinjlim_i H_q(X, X - A_i) \xrightarrow{\cong} H_q(X, X - A)\,.$$

Proof. Let $\sigma : \Delta^q \to X$ be any q-simplex in $X - A$. The subsets $X - A_i$ form an open cover of $\mathrm{im}(\sigma)$, so by compactness it lies in some single $X - A_i$. This shows that

$$\varinjlim_i S_q(X - A_i) \xrightarrow{\cong} S_q(X - A)\,.$$

Thus

$$\varinjlim_i S_q(X|A_i) \xrightarrow{\cong} S_q(X|A_i)$$

by exactness of direct limit, and the claim then follows for the same reason. □

Lemma 32.5. *Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence of compact subsets in a Hausdorff space X with intersection A. For any open neighborhood U of A there exists i such that $A_i \subseteq U$.*

Proof. A is compact, being a closed subset of the compact Hausdorff space A_1. Since A is the intersection of the A_i, and $A \subseteq U$, the intersection of the decreasing sequence of compact sets $A_i - U$ is empty. Thus by the finite intersection property one of them must be empty; but that says that $A_i \subseteq U$. □

Proof of Proposition 32.3. By Lemma 32.4, $H_q(M|A) = 0$ for $q > n$. In dimension n, we contemplate the commutative diagram

$$\begin{array}{ccc}
\varinjlim_i H_n(M|A_i) & \xrightarrow{\cong} & H_n(M|A) \\
\downarrow{\cong} & & \downarrow \\
\varinjlim_i \Gamma(A_i; o_M) & \xrightarrow{\cong} & \Gamma(A; o_M)\,.
\end{array}$$

The top map an isomorphism by Lemma 32.4.

To see that the bottom map is an isomorphism, we'll verify the two conditions for a map to be a direct limit from Lecture 23. First let x be a section of o_M over A. By compactness, we may cover A by a finite set of opens over each of which o_M is trivial. The section extends over their union U, by unique path lifting. By Lemma 32.5 this open set contains some A_i, and we conclude that any section over A extends to some A_i.

On the other hand, suppose that a section $x \in \Gamma(A_i; o_M)$ vanishes on A. Then it vanishes on some open set containing A, again by unique path lifting and local triviality. Some A_j lies in that open set, again by Lemma 32.5. We may assume that $j \geq i$, and conclude that x already vanishes on A_j. □

Proof of Theorem 32.1. There are five steps. In describing them, we will call a subset of M "Euclidean" if it lies inside some open set homeomorphic to $\mathbf{R}^n$.

(1) $M = \mathbf{R}^n$, A a compact convex subset.
(2) $M = \mathbf{R}^n$, A a finite union of compact convex subsets.
(3) $M = \mathbf{R}^n$, A any compact subset.
(4) M arbitrary, A a finite union of compact Euclidean subsets.
(5) M arbitrary, A an arbitrary compact subset.

Notes on the proofs: (1) To be clear, "convex" implies nonempty. By translating A, we may assume that $0 \in A$. The compact subset A lies in some disk, and by a homothety we may assume that the disk is the unit disk D^n. Then we claim that the inclusion $i : S^{n-1} \to \mathbf{R}^n - A$ is a deformation retract. A retraction is given by $r(x) = x/||x||$, and a homotopy from ir to the identity is given by

$$h(x,t) = \left(t + \frac{1-t}{||x||}\right) x\,.$$

It follows that $H_q(\mathbf{R}^n, \mathbf{R}^n - A) \cong H_q(\mathbf{R}^n, \mathbf{R}^n - D^n)$ for all q. This group is zero for $q > n$. In dimension n, note that restricting to the origin gives an isomorphism $H_n(\mathbf{R}^n, \mathbf{R}^n - D^n) \to H_n(\mathbf{R}^n, \mathbf{R}^n - 0)$ since $\mathbf{R}^n - D$ is a deformation retract of $\mathbf{R}^n - 0$. The local system $o_{\mathbf{R}^n}$ is trivial, since $\mathbf{R}^n$ is simply connected, so restricting to the origin gives an isomorphism $\Gamma(D^n, o_{\mathbf{R}^n}) \to H_n(\mathbf{R}^n, \mathbf{R}^n - 0)$. This implies that $j_{D^n} : H_n(\mathbf{R}^n, \mathbf{R}^n - D^n) \to \Gamma(D^n, o_{\mathbf{R}^n})$ is an isomorphism. The restriction $\Gamma(D^n, o_{\mathbf{R}^n}) \to \Gamma(A, o_{\mathbf{R}^n})$ is also an isomorphism, since $A \to D^n$ is a deformation retract. So by the commutative diagram

$$\begin{array}{ccc} H_n(\mathbf{R}^n, \mathbf{R}^n - D^n) & \xrightarrow{\cong} & H_n(\mathbf{R}^n, \mathbf{R}^n - A) \\ \downarrow{\scriptstyle j_{D^n}} & & \downarrow{\scriptstyle j_A} \\ \Gamma(D^n, o_{\mathbf{R}^n}) & \longrightarrow & \Gamma(A, o_{\mathbf{R}^n}) \end{array}$$

we find that $j_A : H_n(\mathbf{R}^n, \mathbf{R}^n - A) \to \Gamma(A; o_{\mathbf{R}^n})$ is an isomorphism.

(2) by Proposition 32.2.

(3) For each $j \geq 1$, let C_j be a finite subset of A such that

$$A \subseteq \bigcup_{x \in C_j} B_{1/j}(x)\,.$$

Since any intersection of convex sets is either empty or convex,

$$A_k = \bigcap_{j=1}^{k} \bigcup_{x \in C_j} B_{1/j}(x)$$

is a union of finitely many convex sets, and since A is closed it is the intersection of this decreasing family. So the result follows from (1), (2), and Proposition 32.3.

(4) by (3) and (2).

(5) Cover A by finitely many open subsets that embed in Euclidean opens as open disks with compact closures. Their closures then form a finite cover by closed Euclidean disks D_i in Euclidean opens U_i. For each i, excise the closed subset $M - U_i$ to see that

$$H_q(M, M - A \cap D_i) \cong H_q(U_i, U_i - A \cap D_i) \cong H_q(\mathbf{R}^n, \mathbf{R}^n - A \cap D_i)\,.$$

By (4), the theorem holds for each of these. Each intersection $(A \cap D_i) \cap (A \cap D_j)$ is again a compact Euclidean subset, so the result holds for them by excision as well. The result then follows by (1). $\square$

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