33 A plethora of products

We are now heading towards a statement of Poincaré duality.

Recall that we have the Kronecker pairing

$$\langle -,-\rangle : H^p(X;R)\otimes H_p(X;R)\to R\,.$$

It's obviously not "natural," because H^p is contravariant while homology is covariant. But given $f: X\to Y$, $b\in H^p(Y)$, and $x\in H_p(X)$, we can ask: How does $\langle f^*b,x\rangle$ relate to $\langle b,f_*x\rangle$?

Claim 33.1. $\langle f^*b,x\rangle = \langle b,f_*x\rangle$.

Proof. This is easy! I find it useful to write out diagrams to show where things are. We're going to work on the chain level.

$$\begin{array}{ccc}
\mathrm{Hom}(S_p(Y),R)\otimes S_p(X) & \xrightarrow{1\otimes f_*} & \mathrm{Hom}(S_p(Y),R)\otimes S_p(Y) \\
\Big\downarrow{\scriptstyle f^*\otimes 1} & & \Big\downarrow{\scriptstyle \langle -,-\rangle} \\
\mathrm{Hom}(S_p(X),R)\otimes S_p(X) & \xrightarrow{\langle -,-\rangle} & R
\end{array}$$

We want this diagram to commute. Suppose $[\beta]=b$ and $[\xi]=x$. Then going to the right and then down gives

$$\beta\otimes\xi\mapsto\beta\otimes f_*(\xi)\mapsto\beta(f_*\xi)\,.$$

The other way gives

$$\beta\otimes\xi\mapsto f^*(\beta)\otimes\xi=(\beta\circ f_*)\otimes\xi\mapsto(\beta\circ f_*)(\xi)\,.$$

This is exactly $\beta(f_*\xi)$. □

There's actually another product in play here:

$$\mu : H(C_*)\otimes H(D_*)\to H(C_*\otimes D_*)$$

given by $[c]\otimes[d]\mapsto[c\otimes d]$. I used it to pass from the chain level computation we did to the homology statement.

We also have the two cross products:

$$\times : H_p(X)\otimes H_q(Y)\to H_{p+q}(X\times Y)$$

and

$$\times : H^p(X)\otimes H^q(Y)\to H^{p+q}(X\times Y)\,.$$

You might think this is fishy because both maps are in the same direction. But it's OK, because we used different things to make these constructions: the chain-level cross product (or Eilenberg-Zilber map) for homology and the Alexander-Whitney map for cohomology. Still, they're related:

Lemma 33.2. *Let $a\in H^p(X), b\in H^q(Y), x\in H_p(X), y\in H_q(Y)$. Then:*

$$\langle a\times b, x\times y\rangle = (-1)^{|x|\cdot|b|}\langle a,x\rangle\langle b,y\rangle\,.$$

Proof. Look at the chain-level cross product and the Alexander-Whitney maps:

$$\times : S_*(X) \otimes S_*(Y) \leftrightarrows S_*(X \times Y) : \alpha$$

They are inverse isomorphisms in dimension 0, and both sides are projective resolutions with respect to the models (Δ^p, Δ^q); so by acyclic models they are natural chain homotopy inverses.

Say $[f] = a, [g] = b, [\xi] = x, [\eta] = y$. Write fg for the composite

$$S_p(X) \otimes S_q(Y) \xrightarrow{\times} S_{p+q}(X \times Y) \xrightarrow{f \otimes g} R \otimes R \to R\,.$$

Then:

$$(f \times g)(\xi \times \eta) = (fg)\alpha(\xi \times \eta) \simeq (fg)(\xi \otimes \eta) = (-1)^{pq} f(\xi)g(\eta)\,.$$

□

We can use this to prove a restricted form of the Künneth theorem in cohomology.

Theorem 33.3. *Let R be a PID. Assume that $H_p(X)$ is a finitely generated free R-module for all p. Then*

$$\times : H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$$

is an isomorphism.

Proof. Write $M^\vee$ for the linear dual of an R-module M. By our assumption about $H_p(X)$, the map

$$H_p(X)^\vee \otimes H_q(Y)^\vee \to (H_p(X) \otimes H_q(Y))^\vee \ ,$$

sending $f \otimes g$ to $(x \otimes y \mapsto (-1)^{pq} f(x)g(y))$, is an isomorphism. The homology Künneth theorem guarantees that the bottom map in the following diagram is an isomorphism.

$$\begin{array}{ccccc}
\bigoplus_{p+q=n} H^p(X) \otimes H^q(Y) & & \xrightarrow{\quad\times\quad} & & H^n(X \times Y) \\
\downarrow \cong & & & & \downarrow \cong \\
\bigoplus_{p+q=n} H_p(X)^\vee \otimes H_q(Y)^\vee & \xrightarrow{\cong} & \left(\bigoplus_{p+q=n} H_p(X) \otimes H_q(Y)\right)^\vee & \xleftarrow{\cong} & H_n(X \times Y)^\vee
\end{array}$$

Commutativity of this diagram is exactly the content of Lemma 33.2. □

We saw before that $\times$ is an algebra map, so under the conditions of the theorem it is an isomorphism of algebras. You do need some finiteness assumption, even if you are working over a field. For example let T be an infinite set, regarded as a space with the discrete topology. Then $H^0(T;R) = \mathrm{Map}(T,R)$. But

$$\mathrm{Map}(T,R) \otimes \mathrm{Map}(T,R) \to \mathrm{Map}(T \times T, R)$$

sending $f \otimes g$ to $(s,t) \to f(s)g(t)$ is not surjective; the characteristic function of the diagonal is not in the image, for example (unless $R = 0$).

There are more products around. For example, there is a map

$$H^p(Y) \otimes H^q(X,A) \to H^{p+q}(Y \times X, Y \times A)\,.$$

Constructing this is on your homework. Suppose $Y = X$. Then I get

$$\cup : H^*(X) \otimes H^*(X,A) \to H^*(X \times X, X \times A) \xrightarrow{\Delta^*} H^*(X,A)$$

where $\Delta : (X, A) \to (X \times X, X \times A)$ is the "relative diagonal." This *relative cup product* makes $H^*(X, A)$ into a module over the graded algebra $H^*(X)$. The relative cohomology is *not* a ring – it doesn't have a unit, for example – but it is a module. And the long exact sequence of the pair is a sequence of $H^*(X)$-modules.

I want to introduce you to one more product, one that will enter into our expression of Poincaré duality. This is the *cap product*. What can I do with $S^p(X) \otimes S_n(X)$? Well, I can form the composite:

$$\cap : S^p(X) \otimes S_n(X) \xrightarrow{1\times(\alpha\circ\Delta_*)} S^p(X) \otimes S_p(X) \otimes S_{n-p}(X) \xrightarrow{\langle -,-\rangle\otimes 1} S_{n-p}(X)$$

Using our explicit formula for α, we can write:

$$\cap : \beta \otimes \sigma \mapsto \beta \otimes (\sigma \circ \alpha_p) \otimes (\sigma \circ \omega_q) \mapsto (\beta(\sigma \circ \alpha_p))\,(\sigma \circ \omega_q)$$

We are evaluating the cochain on *part of* the chain, leaving a lower dimensional chain left over.

This composite is a chain map, and so induces a map in homology:

$$\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X)\,.$$

Notice how the dimensions work. Long ago a bad choice was made: If cohomology were graded with negative integers, the way the gradations work here would look better.

There are also two slant products. Maybe I won't talk about them. In the next lecture, I'll check a few things about cap products, and then get into the machinery of Poincaré duality.