34 Cap product and “Cech” cohomology

We have a few more things to say about the cap product, and will then use it to give a statement of Poincaré duality.

Proposition 34.1. *The cap product enjoys the following properties.*
(1) $(a \cup b) \cap x = a \cap (b \cap x)$ *and* $1 \cap x = x$*:* $H_*(X)$ *is a module for* $H^*(X)$*.*
(2) Given a map $f : X \to Y$*,* $b \in H^p(Y)$*, and* $x \in H_n(X)$*,*

$$f_*(f^*(b) \cap x) = b \cap f_*(x)\,.$$

(3) Let $\epsilon : H_*(X) \to R$ *be the augmentation. Then*

$$\varepsilon(b \cap x) = \langle b, x \rangle\,.$$

(4) Cap and cup are adjoint:

$$\langle a \cap b, x \rangle = \langle a, b \cap x \rangle\,.$$

Proof. (1) Easy.
(2) Let β be a cocycle representing b, and σ an n-simplex in X. Then

$$\begin{aligned} f_*(f^*(\beta) \cap \sigma) &= f_*((f^*(\beta)(\sigma \circ \alpha_p)) \cdot (\sigma \circ \omega_q)) \\ &= f_*(\beta(f \circ \sigma \circ \alpha_p) \cdot (\sigma \circ \omega)) \\ &= \beta(f \circ \sigma \circ \alpha_p) \cdot f_*(\sigma \circ \omega_q) \\ &= \beta(f \circ \sigma \circ \alpha_p) \cdot (f \circ \sigma \circ \omega_q) \\ &= \beta \cap f_*(\sigma) \end{aligned}$$

This formula goes by many names: the "projection formula," or "Frobenius reciprocity."
(3) We get zero unless $p = n$. Again let $\sigma \in \mathrm{Sin}_n(X)$, and compute:

$$\varepsilon(\beta \cap \sigma) = \varepsilon(\beta(\sigma) \cdot c^0_{\sigma(n)}) = \beta(\sigma)\varepsilon(c^0_{\sigma(n)}) = \beta(\sigma) = \langle \beta, \sigma \rangle \,.$$

□

Here now is a statement of Poincaré duality. It deals with the homological structure of compact topological manifolds. We recall the notion of an orientation, and Theorem 31.9 asserting the existence of a fundamental class $[M] \in H_n(M; R)$ in a compact R-oriented n-manifold.

Theorem 34.2 (Poincaré duality)**.** *Let M be a topological n-manifold that is compact and oriented with respect to a PID R. Then there is a unique class $[M] \in H_n(M; R)$ that restricts to the orientation class in $H_n(M, M - a; R)$ for every $a \in M$. It has the property that*

$$- \cap [M] : H^p(M; R) \to H_q(M; R) \,, \quad p + q = n \,,$$

is an isomorphism for all p.

You might want to go back to Lecture 25 and verify that $\mathbf{RP}^3 \times \mathbf{RP}^3$ satisfies this theorem.

Our proof of Poincaré duality will be by induction. In order to make the induction go we will prove a substantially more general theorem, one that involves relative homology and cohomology. So we begin by understanding how the cap product behaves in relative homology.

Suppose $A \subseteq X$ is a subspace. We have:

$$\begin{array}{ccccc}
0 & & & & 0 \\
\downarrow & & & & \downarrow \\
S^p(X) \otimes S_n(A) & \xrightarrow{i^* \otimes 1} & S^p(A) \otimes S_n(A) & \xrightarrow{\cap} & S_q(A) \\
\downarrow{\scriptstyle 1 \otimes i_*} & & & & \downarrow{\scriptstyle i_*} \\
S^p(X) \otimes S_n(X) & & \xrightarrow{\quad\cap\quad} & & S_q(X) \\
\downarrow & & & & \downarrow \\
S^p(X) \otimes S_n(X, A) & & \dashrightarrow & & S_q(X, A) \\
\downarrow & & & & \downarrow \\
0 & & & & 0
\end{array}$$

The left sequence is exact because $0 \to S_n(A) \to S_n(X) \to S_n(X, A) \to 0$ splits and tensoring with $S^p(X)$ (which is not free!) therefore leaves it exact. The solid arrow diagram commutes precisely by the chain-level projection formula. There is therefore a uniquely defined map on cokernels.

This chain map yields the *relative cap product*

$$\cap : H^p(X) \otimes H_n(X, A) \to H_q(X, A)$$

It renders $H_*(X, A)$ a module for the graded algebra $H^*(X)$.

I want to come back to an old question, about the significance of relative homology. Suppose that $K \subseteq X$ is a subspace, and consider the relative homology $H_*(X, X - K)$. Since the complement of $X - K$ in X is K, these groups should be regarded as giving information about K. If I enlarge

K, I make $X - K$ smaller: $K \subseteq L$ induces $H_*(X, X - L) \to H_*(X - K)$; the relative homology is *contravariant* in the variable K (regarded as an object of the poset of subspaces of X).

Excision gives insight into how $H_*(X, X - K)$ depends on K. Suppose $K \subseteq U \subseteq X$ with $\overline{K} \subseteq \mathrm{Int}(U)$. To simplify things, let's just suppose that K is closed and U is open. Then $X - U$ is closed, $X - K$ is open, and $X - U \subseteq X - K$, so excision asserts that the inclusion map

$$H_*(U, U - K) \to H_*(X, X - K)$$

is an isomorphism.

The cap product puts some structure on $H_*(X, X - K)$: it's a module over $H^*(X)$. But we can do better! We just decided that $H_*(X, X - K) = H_*(U, U - K)$, so the $H^*(X)$ action factors through an action by $H^*(U)$, for any open set U containing K. How does this refined action change when I decrease U?

Lemma 34.3. *Let $K \subseteq V \subseteq U \subseteq X$, with K closed and U, V open. Then:*

$$\begin{array}{ccc} H^p(U) \otimes H_n(X, X-K) & & \\ \downarrow{\scriptstyle i^* \otimes 1} & \searrow^{\cap} & \\ & & H_q(X, X-K) \\ & \nearrow_{\cap} & \\ H^p(V) \otimes H_n(X, X-K) & & \end{array}$$

commutes.

Proof. This is just the projection formula again! □

Let $\mathcal{U}_K$ be the set of open neighborhoods of K in X. It is partially ordered by reverse inclusion. This poset is directed, since the intersection of two opens is open. By the lemma, $H^p : \mathcal{U}_K \to \mathbf{Ab}$ is a directed system.

Definition 34.4. The *Čech cohomology* of K is

$$\check{H}^p(K) = \varinjlim_{U \in \mathcal{U}_K} H^p(U)\,.$$

I apologize for this bad notation; its possible dependence on the way K is sitting in X is not recorded. The maps in this directed systen are all maps of graded algebras, so the direct limit is naturally a commutative graded algebra. Since tensor product commutes with direct limits, we now get a cap product pairing

$$\cap : \check{H}^p(K) \otimes H_n(X, X - K) \to H_q(X, X - K)$$

satifying the expected properties. This is the best you can do. It's the natural structure that this relative homology has: $H_*(X, X - K)$ is a module over $\check{H}^*(K)$.

There are compatible restriction maps $H^p(U) \to H^p(K)$, so there is a natural map

$$\check{H}^*(K) \to H^*(K)\,.$$

This map is often an isomorphism. Suppose $K \subseteq X$ satisfies the following "regular neighborhood" condition: For every open $U \supseteq K$, there exists an open V with $U \supseteq V \supseteq K$ such that $K \hookrightarrow V$ is a homotopy equivalence (or actually just a homology isomorphism).

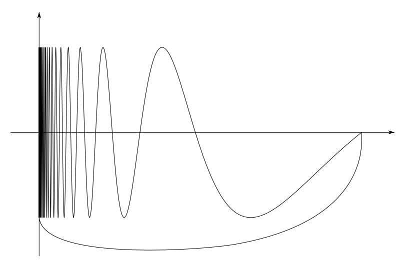
Lemma 34.5. *Under these conditions, $\check{H}^*(K) \to H^*(K)$ is an isomorphism.*

Proof. We will check that the map to $H^p(K)$ satisfies the conditions we established in Lecture 23 to be a direct limit.

So let $x \in H^p(K)$. Let U be a neighborood of K in X such that $H^p(U) \to H^p(K)$ is an isomorphism. Then indeed x is in the image of $H^p(U)$.

Then let U be a neighborhood of K and let $x \in H^p(U)$ restrict to 0 in $H^p(K)$. Let V be a sub-neighborood such that $H^p(V) \to H^p(K)$ is an isomorphism. Then x restricts to 0 in $H^p(V)$. □

On the other hand, here's an example that distinguishes $\check{H}^*$ from H^*. This is a famous example. The "topologist's sine curve" is the subspace of $\mathbf{R}^2$ defined as follows. It is union of three subsets, A, B, and C. A is the graph of $\sin(\pi/x)$ where $0 < x < 1$. B is the interval $0 \times [-1, 1]$. C is a continuous curve from $(0, -1)$ to $(1, 0)$ and meeting $A \cup B$ only at its endpoints. This is a counterexample for a lot of things; you've probably seen it in 18.901.



What is the singular homology of the topologist's sine curve? Use Mayer-Vietoris! I can choose V to be some connected portion of the continuous curve from $(0, -1)$ to $(1, 0)$, and U to contain the rest of the space in a way that intersects V in two open intervals. Then V is contractible, and U is made up of two contractible connected components. (This space is not locally path connected, and one of these path components is not closed.)

The Mayer-Vietoris sequence looks like

$$0 \to H_1(X) \xrightarrow{\partial} H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0\,.$$

The two path components of $U \cap V$ do not become connected in U, so $\partial = 0$ and we find that $\varepsilon : H_*(X) \xrightarrow{\cong} H_*(*)$ and hence $H^*(X) \cong H^*(*)$.

How about $\check{H}^*$? Let $X \subset U$ be an open neighborhood. The interval $0 \times [-1, 1]$ has an ϵ-neighborhood, for some small ϵ, that's contained in U. This implies that there exists a neighborhood $X \subseteq V \subseteq U$ such that $V \simeq S^1$. This implies that

$$\varinjlim_{U \in \mathcal{U}_X} H^*(U) \cong H^*(S^1)$$

by a cofinality argument that we will detail later. So $\check{H}^*(X) \neq H^*(X)$.

Nevertheless, under quite general conditions the Čech cohomology of a compact Hausdorff space is a topological invariant. The Čech construction forms a limit over open covers of the cohomology of the nerve of the cover. It is a topological invariant by construction.

Theorem 34.6. *Let X be a compact subset of some Euclidean space. If there is an open neighborhood of which it is a retract, then $\check{H}^*(X; R)$ is canonically isomorphic to the cohomology defined using the Čech construction, and is therefore independent of the embedding into Euclidean space.*

See Dold's beautiful book [3] for this and other topics discussed in this chapter.

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