35 Cech cohomology as a cohomology theory

Let X be any space, and let $K \subseteq X$ be a closed subspace. We've defined the Čech cohomology of K as the direct limit of $H^*(U)$ as U ranges over the poset $\mathcal{U}_K$ of open neighborhoods of K. This often coincides with $H^*(K)$ but will not be the same in general. Nevertheless it behaves like a cohomology theory. To expand on this claim, we should begin by defining a relative version.

Suppose $L \subseteq K$ is a pair of closed subsets of a space X. Let (U, V) be a "neighborhood pair" for (K, L):

$$\begin{array}{ccc} L & \subseteq & K \\ \cap\!| & & \cap\!| \\ V & \subseteq & U \end{array}$$

with U and V open. These again form a directed set $\mathcal{U}_{K,L}$, with partial order given by reverse inclusion of pairs. Then define

$$\check{H}^p(K, L) = \varinjlim_{(U,V) \in \mathcal{U}_{K,L}} H^p(U, V)\,.$$

We will want to verify versions of the Eilenberg-Steenrod axioms for these functors. For a start, I have to explain how maps induce maps.

Let $\mathcal{I}$ be a directed set and $A : \mathcal{I} \to \mathbf{Ab}$ a functor. If we have an order-preserving map – a functor – $\varphi : \mathcal{J} \to \mathcal{I}$ from another directed set, we get $A\varphi : \mathcal{J} \to \mathbf{Ab}$; so $(A\varphi)_j = A_{\varphi(j)}$. I can form two direct limits: $\varinjlim_{\mathcal{J}} A\varphi$ and $\varinjlim_{\mathcal{I}} A$. I claim that they are related by a map

$$\varinjlim_{\mathcal{J}} A\varphi \to \varinjlim_{\mathcal{I}} A\,.$$

Using the universal property of direct limits, we need to come up with compatible maps $f_j : A_{\varphi(j)} \to \varinjlim_{\mathcal{I}} A$. We have compatible maps $\mathrm{in}_i : A_i \to \varinjlim_{\mathcal{I}} A$ for $i \in \mathcal{I}$, so we can take $f_j = \mathrm{in}_{\varphi(j)}$.

These maps are compatible under composition of order-preserving maps.

Example 35.1. A closed inclusion $i : K \supseteq L$ induces an order-preserving map $\varphi : \mathcal{U}_K \to \mathcal{U}_L$. The functor $H^p : \mathcal{U}_K \to \mathbf{Ab}$ restricts to $H^p : \mathcal{U}_L \to \mathbf{Ab}$, so we get maps

$$\varinjlim_{\mathcal{U}_K} H^p = \varinjlim_{\mathcal{U}_K} H^p\varphi \to \varinjlim_{\mathcal{U}_L} H^p\,.$$

i.e.

$$i^* : \check{H}^p(K) \to \check{H}^p(L)\,.$$

This makes $\check{H}^p$ into a contravariant functor on the partially ordered set of closed subsets of X.

I can do the same thing for relative cohomology, and get the maps involved in the following two theorems, whose proofs will come in due course.

Theorem 35.2 (Long exact sequence)**.** *Let (K, L) be a closed pair in X. There is a long exact sequence*

$$\cdots \to \check{H}^p(K, L) \to \check{H}^p(K) \to \check{H}^p(L) \xrightarrow{\delta} \check{H}^{p+1}(K, L) \to \cdots$$

that is natural in the pair.

Theorem 35.3 (Excision)**.** *Suppose A and B are closed subsets of a normal space, or compact subsets of a Hausdorff space. Then the map*

$$\check{H}^p(A \cup B, A) \xrightarrow{\cong} \check{H}^p(B, A \cap B)$$

induced by the inclusion is an isomorphism.

Each of these theorems relates direct limits defined over different directed sets. To prove them, I will want to rewrite the various direct limits as direct limits over the same directed set. This raises the following ...

Question 35.4. When does $\varphi : \mathcal{J} \to \mathcal{I}$ induce an isomorphism $\varinjlim_{\mathcal{J}} A\varphi \to \varinjlim_{\mathcal{I}} A$?

This is a lot like taking a sequence and a subsequence and asking when they have the same limit. There's a cofinality condition in analysis, that has a similar expression here.

Definition 35.5. $\varphi : \mathcal{J} \to \mathcal{I}$ is *cofinal* if for all $i \in \mathcal{I}$, there exists $j \in \mathcal{J}$ such that $i \leq \varphi(j)$.

Example 35.6. Any surjective order-preserving map is cofinal.

For another example, let $(\mathbb{N}_{>0}, <)$ be the positive integers with their ususal order, and $(\mathbb{N}_{>0}, |)$ the same set but with the divisiblity order. There is an order-preserving map $\varphi : (\mathbb{N}_{>0}, <) \to (\mathbb{N}_{>0}, |)$ given by $n \mapsto n!$. This map is far from surjective, but any integer n divides some factorial (n divides $n!$, for example), so φ is cofinal. We claimed that both these systems produce $\mathbf{Q}$ as direct limit.

Lemma 35.7. *If $\varphi : \mathcal{J} \to \mathcal{I}$ is cofinal then $\varinjlim_{\mathcal{J}} A\varphi \to \varinjlim_{\mathcal{I}} A$ is an isomorphism.*

Proof. Check that $\{A_{\varphi(j)} \to \varinjlim_{\mathcal{I}} A\}$ satisfies the necessary and sufficient conditions to be $\varinjlim_{\mathcal{J}} A\varphi$.

1. For each $a \in \varinjlim_{\mathcal{I}} A$ there exists $j \in \mathcal{J}$ and $a_j \in A_{\varphi(j)}$ such that $a_j \mapsto a$: We know that there exists some $i \in \mathcal{I}$ and $a_i \in A$ such that $a_i \mapsto a$. Pick j such that $i \leq \varphi(j)$. Then $a_i \mapsto a_{\varphi(j)}$, and by compatibility we get $a_{\varphi(j)} \mapsto a$.

2. Suppose $a \in A_{\varphi(j)}$ maps to $0 \in \varinjlim_{\mathcal{I}} A$. Then there is some $i \in \mathcal{I}$ such that $\varphi(j) \leq i$ and $a \mapsto 0$ in A_i. But then there is $j' \in \mathcal{J}$ such that $i \leq \varphi(j')$, and $a \mapsto 0 \in A_{\varphi(j')}$ as well.

□

Proof of Theorem 35.2, the long exact sequence. Let (K, L) be a closed pair in the space X. We have

$$\check{H}^p(K, L) = \varinjlim_{(U,V) \in \mathcal{U}_{K,L}} H^p(U, V)\,, \quad \check{H}^p(K) = \varinjlim_{U \in \mathcal{U}_K} H^p(U)\,, \quad \text{and} \quad \check{H}^p(L) = \varinjlim_{V \in \mathcal{V}_L} H^p(V)\,.$$

We can rewrite the entire sequence as the direct limit of a directed system of exact sequences indexed by $\mathcal{U}_{K,L}$, since the order-preserving maps

$$\mathcal{U}_K \leftarrow \mathcal{U}_{K,L} \to \mathcal{U}_L$$

$$U \leftarrow\!\shortmid (U, V) \mapsto V$$

are both surjective and hence cofinal. So the long exact sequence of a pair in Čech cohomology is the direct limit of the system of long exact sequences of the neighborhood pairs (U, V) and so is exact. □

The proof of the excision theorem depends upon another pair of cofinalities.

Lemma 35.8. *Assume that X is a normal space and A, B closed subsets, or that X is a Hausdorff space and A, B compact subsets. Then the order-preserving maps*

$$\mathcal{U}_{(A\cup B,B)} \leftarrow \mathcal{U}_A \times \mathcal{U}_B \rightarrow \mathcal{U}_{(A,A\cap B)}$$

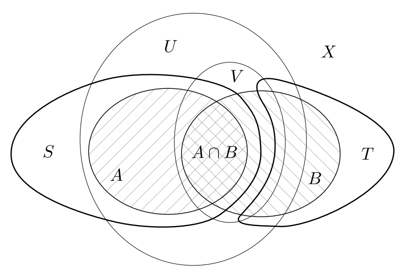
given by

$$(W \cup Y, Y) \leftarrow\!\shortmid (W, Y) \mapsto (W, W \cap Y)$$

are both cofinal.

Proof. The left map is surjective, because if $(U, V) \in \mathcal{U}_{A\cup B,B}$ then $U \in \mathcal{U}_A$, $V \in \mathcal{U}_B$, and $(U, V) = (U \cup V, V)$.

To see that the right map is cofinal, start with $(U, V) \in \mathcal{U}_{A,A\cap B}$.



Note that A is disjoint from $B \cap (X - V)$, so by normality, or compactness in a Hausdorff space, there exist non-intersecting open sets S and T with $A \subseteq S$ and $B \cap (X - V) \subseteq T$. Then take $W = U \cap S \in \mathcal{U}_A$ and $Y = V \cup T \in \mathcal{U}_B$, and observe that $W \cap Y = V \cap S$ and so $(W, W \cap Y) \subseteq (U, V)$. □

Proof of Theorem 35.3. Combine Lemma 35.8 with excision for singular cohomology:

$$\begin{array}{ccc}
\varinjlim_{(W,Y)\in\mathcal{U}_A\times\mathcal{U}_B} H^p(W \cup Y, Y) & \xrightarrow{\cong} & \varinjlim_{\mathcal{U}_A\times\mathcal{U}_B} H^p(W, W \cap Y) \\
\downarrow \cong & & \downarrow \cong \\
\varinjlim_{(U,V)\in\mathcal{U}_{A\cup B,B}} H^p(U, V) & \longrightarrow & \varinjlim_{(U,V)\in\mathcal{U}_{A,A\cap B}} H^p(U, V) \\
\| & & \| \\
\check{H}^p(A \cup B, B) & \longrightarrow & \check{H}^p(A, A \cap B)
\end{array}$$

□

The Mayer-Vietoris long exact sequence is a consequence of these two results.

Corollary 35.9 (Mayer-Vietoris)**.** *Suppose A and B are closed subsets of a normal space, or compact subsets of a Hausdorff space. There is a natural long exact sequence:*

$$\cdots \rightarrow \check{H}^{p-1}(A \cup B) \rightarrow \check{H}^{p-1}(A) \oplus \check{H}^p(B) \rightarrow \check{H}^{p-1}(A \cap B) \rightarrow H^p(A \cup B) \rightarrow \cdots .$$

Proof. Apply Lemma 11.6 to the ladder

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & \check{H}^{p-1}(A\cup B) & \longrightarrow & \check{H}^{p-1}(B) & \longrightarrow & \check{H}^{p}(A\cup B, B) & \longrightarrow & \check{H}^{p}(A\cup B) & \longrightarrow & \check{H}^{p}(B) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow\cong & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \check{H}^{p-1}(A) & \longrightarrow & \check{H}^{p-1}(A\cap B) & \longrightarrow & \check{H}^{p}(A, A\cap B) & \longrightarrow & \check{H}^{p}(A) & \longrightarrow & \check{H}^{p}(A\cap B) & \longrightarrow & \cdots\,.
\end{array}$$

□