38 Applications

Today we harvest consequences of Poincaré duality. We'll use the form

Theorem 38.1. *Let M be an n-manifold and K a compact subset. An R-orientation along K determines a fundamental class $[M]_K \in H_n(M, M-K)$, and capping gives an isomorphism:*

$$\cap[M]_K : \check{H}^{n-q}(K;R) \xrightarrow{\cong} H_q(M, M-K;R)\,.$$

Corollary 38.2. *$\check{H}^p(K;R) = 0$ for $p > n$.*

We can contrast this with singular (co)homology. Here's an example:

Example 38.3 (Barratt-Milnor, [1])**.** A two-dimensional version K of the Hawaiian earring, i.e., nested spheres all tangent to a point whose radii are going to zero. What they proved is that $H_q(K;\mathbf{Q})$ is uncountable for every $q > 1$. But Čech cohomology is much more well-behaved.

Theorem 38.4 (Alexander duality)**.** *For any compact subset K of $\mathbf{R}^n$, the composite*

$$\check{H}^{n-q}(K;R) \xrightarrow{\cap[\mathbf{R}^n]_K} H_q(\mathbf{R}^n, \mathbf{R}^n - K;R) \xrightarrow{\partial} \widetilde{H}_{q-1}(\mathbf{R}^n - K;R)$$

is an isomorphism.

Proof. $\widetilde{H}^*(\mathbf{R}^n;R) = 0$. □

This is extremely useful! For example

Corollary 38.5. *If K is a compact subset of $\mathbf{R}^n$ then $\check{H}^n(K;R) = 0$.*

Corollary 38.6. *The complement of a knot in S^3 is a homology circle.*

Example 38.7. Take the case $q = 1$:

$$\check{H}^{n-1}(K;R) \xrightarrow{\cong} \widetilde{H}_0(\mathbf{R}^n - K;R) = \ker(\varepsilon : R\pi_0(\mathbf{R}^n - K) \to R)\,.$$

The augmentation is a split surjection, so this is a free R-module. This shows, for example, that $\mathbf{RP}^2$ can't be embedded in $\mathbf{R}^3$ – at least not with a regular neighborhood.

If we take $n = 2$ and suppose that $\check{H}^*(K) = H^*(S^1)$, we find that the complement of K has *two* path components. This is the *Jordan Curve Theorem.*

There is a useful purely cohomological consequence of Poincaré duality, obtained by combining it with the universal coeffient theorem

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_{q-1}(X), \mathbf{Z}) \to H^q(X) \to \mathrm{Hom}(H_q(X), \mathbf{Z}) \to 0\,.$$

First, note that $\mathrm{Hom}(H_q(X), \mathbf{Z})$ is always torsion-free. If I assume that $H_{q-1}(X)$ is finitely generated, then $\mathrm{Ext}^1_{\mathbf{Z}}(H_{q-1}(X), \mathbf{Z})$ is a finite abelian group. So the UCT is providing the short exact sequence

$$0 \to \mathrm{tors}H^q(X) \to H^q(X) \to H^q(X)/\mathrm{tors} \to 0$$

– that is,

$$H^q(X)/\mathrm{tors} \xrightarrow{\cong} \mathrm{Hom}(H_q(X)/\mathrm{tors}, \mathbf{Z})\,.$$

That is to say, the Kronecker pairing descends to a perfect pairing

$$\frac{H^q(X)}{\text{tors}} \otimes \frac{H_q(X)}{\text{tors}} \to \mathbf{Z}\,.$$

Let's combine this with Poincaré duality. Let $X = M$ be a compact oriented n-manifold, so that

$$\cap[M] : H^{n-q}(M) \xrightarrow{\cong} H_q(M)\,.$$

We get a perfect pairing

$$\frac{H^q(X)}{\text{tors}} \otimes \frac{H^{n-q}(X)}{\text{tors}} \to \mathbf{Z}\,.$$

And what is that pairing? It's given by the composite

$$\begin{array}{ccc} H^q(M) \otimes H^{n-q}(M) & \longrightarrow & \mathbf{Z} \\ \big\downarrow {\scriptstyle 1\otimes(-\cap[M])} & \nearrow {\scriptstyle \langle -,- \rangle} & \\ H^q(M) \otimes H_q(M) & & \end{array}$$

and we've seen that

$$\langle a, b \cap [M] \rangle = \langle a \cup b, [M] \rangle$$

We have used $R = \mathbf{Z}$, but the same argument works for any PID – in particular for any field, in which case $\text{tors}V = 0$. We have proven:

Theorem 38.8. *Let R be a PID an M a compact R-oriented n-manifold. Then*

$$a \otimes b \mapsto \langle a \cup b, [M] \rangle$$

induces a perfect pairing (with $p + q = n$)

$$\frac{H^p(M;R)}{\text{tors}} \otimes_R \frac{H^q(M;R)}{\text{tors}} \to R\,.$$

Example 38.9. Complex projective 2-space is a compact 4-manifold, orientable since it is simply connected. It has a cell structure with cells in dimensions 0, 2, and 4, so its homology is $\mathbf{Z}$ in those dimensions and 0 elsewhere, and so the same is true of its cohomology. Up till now the cup product structure has been a mystery. But now we know that

$$H^2(\mathbf{CP}^2) \otimes H^2(\mathbf{CP}^2) \to H^4(\mathbf{CP}^2)$$

is a perfect pairing. So if we write a for a generator of $H^2(\mathbf{CP}^2)$, then $a \cup a = a^2$ is a free generator for $H^4(\mathbf{CP}^2)$. We have discovered that

$$H^*(\mathbf{CP}^2) = \mathbf{Z}[a]/a^3\,.$$

By the way, notice that if we had chosen $-a$ as a generator, we would still produce the same generator for $H^4(\mathbf{CP}^2)$: so there is a preferred orientation, the one whose fundamental class pairs to 1 against a^2.

This calculation shows that while $\mathbf{CP}^2$ and $S^2 \vee S^4$ are both simply connected and have the same homology, they are not homotopy equivalent. This implies that the attaching map $S^3 \to S^2$ for the top cell in $\mathbf{CP}^2$ – the *Hopf map* – is essential.

How about $\mathbf{CP}^3$? It just adds a 6-cell, so now $H^6(\mathbf{CP}^3) \cong \mathbf{Z}$. The pairing $H^2(\mathbf{CP}^3) \otimes H^4(\mathbf{CP}^3) \to H^6(\mathbf{CP}^3)$ is perfect, so we find that a^3 generates $H^6(\mathbf{CP}^3)$. Continuing in this way, we have

$$H^*(\mathbf{CP}^n) = \mathbf{Z}[a]/(a^{n+1})\,.$$

Example 38.10. Exactly the same argument shows that

$$H^*(\mathbf{RP}^n; \mathbf{F}_2) = \mathbf{F}_2[a]/(a^{n+1})$$

where $|a| = 1$.

I'll end with the following application.

Theorem 38.11 (Borsuk-Ulam)**.** *Think of S^n as the unit vectors in $\mathbf{R}^{n+1}$. For any continuous function $f : S^n \to \mathbf{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.*

Proof. Suppose that no such x exists. Then we may define a continuous function $g : S^n \to S^{n-1}$ by

$$g : x \mapsto \frac{f(x) - f(-x)}{||f(x) - f(-x)||} \, .$$

Note that $g(-x) = -g(x)$: g is equivariant with respect to the antipodal action. It descends to a map $\overline{g} : \mathbf{RP}^n \to \mathbf{RP}^{n-1}$.

We claim that $\overline{g}_* : H_1(\mathbf{RP}^n) \to H_1(\mathbf{RP}^{n-1})$ is nontrivial. To see this, pick a basepoint $b \in S^n$ and choose a 1-simplex $\sigma : \Delta^1 \to S^n$ such that $\sigma(e_0) = b$ and $\sigma(e_1) = -b$. The group $H_1(\mathbf{RP}^n)$ is generated by the cycle $p\sigma$. The image of this cycle in $H_1(\mathbf{RP}^{n-1})$ is represented by the loop $gp\sigma$ at $\overline{b} = pb$, which is the image of the 1-simplex $g\sigma$ joining gb to $g(-b) = -g(b)$. The class of this 1-simplex thus generates $H_1(\mathbf{RP}^{n-1})$.

Therefore $\overline{g}$ is nontrivial in $H_1(-; \mathbf{F}_2)$, and hence also in $H^1(-; \mathbf{F}_2)$. Writing a_n for the generator of $H^1(\mathbf{RP}^n; \mathbf{F}_2)$, we must have $a_n = g^* a_{n-1}$, and consequently $a_n^n = (g^* a_{n-1})^n = g^*(a_{n-1}^n)$. But $H^n(\mathbf{RP}^{n-1}; \mathbf{F}_2) = 0$, so $a_{n-1}^n = 0$; while $a_n^n \neq 0$. This is a contradiction. □

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