4 Categorical language

Let Vect_k be the category of vector spaces over a field k, and linear transformations between them. Given a vector space V, you can consider the dual $V^* = \mathrm{Hom}(V, k)$. Does this give us a functor? If you have a linear transformation $f : V \to W$, you get a map $f^* : W^* \to V^*$, so this is like a functor, but the induced map goes the wrong way. This operation does preserve composition and identities, in an appropriate sense. This is an example of a *contravariant functor*.

I'll leave it to you to spell out the definition, but notice that there is a univeral example of a contravariant functor out of a category $\mathcal{C}$: $\mathcal{C} \to \mathcal{C}^{op}$, where $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$, but $\mathcal{C}^{op}(X, Y)$ is declared to be the set $\mathcal{C}(Y, X)$. The identity morphisms remain the same. To describe the composition in $\mathcal{C}^{op}$, I'll write f^{op} for $f \in \mathcal{C}(Y, X)$ regarded as an element of $\mathcal{C}^{op}(X, Y)$; then $f^{op} \circ g^{op} = (g \circ f)^{op}$.

Then a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is the same thing as a ("covariant") functor from $\mathcal{C}^{op}$ to $\mathcal{D}$.

Let $\mathcal{C}$ be a category, and let $Y \in \mathrm{ob}(\mathcal{C})$. We get a map $\mathcal{C}^{op} \to \mathbf{Set}$ that takes $X \mapsto \mathcal{C}(X, Y)$, and takes a map $X \to W$ to the map defined by composition $\mathcal{C}(W, Y) \to \mathcal{C}(X, Y)$. This is called the functor *represented by* Y. It is very important to note that $\mathcal{C}(-, Y)$ is contravariant, while, on the other hand, for any fixed X, $\mathcal{C}(X, -)$ is a covariant functor (and is said to be "corepresentable" by X).

Example 4.1. Recall that the simplex category $\mathbf{\Delta}$ has objects the totally ordered sets $[n] = \{0, 1, \ldots, n\}$, with order preserving maps as morphisms. The "standard simplex" gives us a functor $\Delta\colon \mathbf{\Delta} \to \mathbf{Top}$. Now fix a space X, and consider

$$[n] \mapsto \mathbf{Top}(\Delta^n, X)\,.$$

This gives us a contravariant functor $\mathbf{\Delta} \to \mathbf{Set}$, or a covariant functor $\mathbf{\Delta}^{op} \to \mathbf{Set}$. This functor carries in it all the face and degeneracy maps we discussed earlier, and their compositions. Let us make a definition.

Definition 4.2. Let $\mathcal{C}$ be any category. A *simplicial object* in $\mathcal{C}$ is a functor $K : \mathbf{\Delta}^{op} \to \mathcal{C}$. Simplicial objects in $\mathcal{C}$ form a category with natural transformations as morphisms. Similarly, *semi-simplicial object* in $\mathcal{C}$ is a functor $\mathbf{\Delta}^{op}_{inj} \to \mathcal{C}$,

So the singular functor Sin_* gives a functor from spaces to simplicial sets (and so, by restriction, to semi-simplicial sets).

I want to interject one more bit of categorical language that will often be useful to us.

Definition 4.3. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is a *split epimorphism* ("split epi" for short) if there exists $g : Y \to X$ (called a section or a splitting) such that the composite $Y \xrightarrow{g} X \xrightarrow{f} Y$ is the identity.

Example 4.4. In the category of sets, a map $f : X \to Y$ is a split epimorphism exactly when, for every element of Y there exists some element of X whose image in Y is the original element. So f is surjective. Is every surjective map a split epimorphism? This is equivalent to the axiom of choice! because a section of f is precisely a choice of $x \in f^{-1}(y)$ for every $y \in Y$.

Every categorical definition is accompanied by a "dual" definition.

Definition 4.5. A map $g : Y \to X$ is a *split monomorphism* ("split mono" for short) if there is $f : X \to Y$ such that $f \circ g = 1_Y$.

Example 4.6. Again let $\mathcal{C} = \mathbf{Set}$. Any split monomorphism is an injection: If $y, y' \in Y$, and $g(y) = g(y')$, we want to show that $y = y'$. Apply f, to get $y = f(g(y)) = f(g(y')) = y'$. But the injection $\varnothing \to Y$ is a split monomorphism only if $Y = \varnothing$. So there's an asymmetry in the category of sets.

Lemma 4.7. *A map is an isomorphism if and only if it is both a split epimorphism and a split monomorphism.*

Proof. Easy! □

The importance of these definitions is this: Functors will not in general respect "monomorphisms" or "epimorphisms," but:

Lemma 4.8. *Any functor sends split epis to split epis and split monos to split monos.*

Proof. Apply F to the diagram establishing f as a split epi or mono. □

Example 4.9. Suppose $\mathcal{C} = \mathbf{Ab}$, and you have a split epi $f : A \to B$. Let $g : B \to A$ be a section. We also have the inclusion $i : \ker f \to A$, and hence a map

$$[\, g \quad i \,] : B \oplus \ker f \to A\,.$$

I leave it to you to check that this map is an isomorphism, and to formulate a dual statement.