5 Homotopy, star-shaped regions

We've computed the homology of a point. Let's now compare the homology of a general space X to this example. There's always a unique map $X \to *$: $*$ is a "terminal object" in **Top**. We have an induced map

$$H_n(X) \to H_n(*) = \begin{cases} \mathbf{Z} & n = 0 \\ 0 & \text{otherwise}. \end{cases}$$

Any formal linear combination $c = \sum a_i x_i$ of points of X is a 0-cycle. The map to $*$ sends c to $\sum a_i \in \mathbf{Z}$. This defines the *augmentation* $\epsilon : H_*(X) \to H_*(*)$. If X is nonempty, the map $X \to *$ is split by any choice of point in X, so the augmentation is also split epi. The kernel of ϵ is the *reduced homology* $\widetilde{H}_*(X)$ of X, and we get a canonical splitting

$$H_*(X) \cong \widetilde{H}_*(X) \oplus \mathbf{Z}.$$

Actually, it's useful to extend the definition to the empty space by the following device. Extend the singular chain complex for any space to include $\mathbf{Z}$ in dimension -1, with $d : S_0(X) \to S_{-1}(X)$ given by the augmentation ϵ sending each 0-simplex to $1 \in \mathbf{Z}$. Let's write $\widetilde{S}_*(X)$ for this chain complex, and $\widetilde{H}_*(X)$ for its homology. When $X \neq \varnothing$, ϵ is surjective and you get the same answer as above. But

$$\widetilde{H}_q(\varnothing) = \begin{cases} \mathbf{Z} & \text{for } q = -1 \\ 0 & \text{for } q \neq -1. \end{cases}$$

This convention is not universally accepted, but I find it useful. $\widetilde{H}_*(X)$ is the *reduced homology* of X.

What other spaces have trivial homology? A slightly non-obvious way to reframe the question is this:

> When do two maps $X \to Y$ induce the same map in homology?

For example, when do $1_X : X \to X$ and $X \to * \to X$ induce the same map in homology? If they do, then $\epsilon : H_*(X) \to \mathbf{Z}$ is an isomorphism.

The key idea is that homology is a discrete invariant, so it should be unchanged by deformation. Here's the definition that makes "deformation" precise.

Definition 5.1. Let $f_0, f_1 : X \to Y$ be two maps. A *homotopy* from f_0 to f_1 is a map $h : X \times I \to Y$ (continuous, of course) such that $h(x, 0) = f_0(x)$ and $f(x, 1) = f_1(x)$. We say that f_0 and f_1 are *homotopic*, and that h is a *homotopy* between them. This relation is denoted by $f_0 \simeq f_1$.

Homotopy is an equivalence relation on maps from X to Y. Transitivity follows from the gluing lemma of point set topology. We denote by $[X, Y]$ the set of *homotopy classes* of maps from X to Y. A key result about homology is this:

Theorem 5.2 (Homotopy invariance of homology)**.** *If $f_0 \simeq f_1$, then $H_*(f_0) = H_*(f_1)$: homology cannot distinguish between homotopic maps.*

Suppose I have two maps $f_0, f_1 : X \to Y$ with a homotopy $h : f_0 \simeq f_1$, and a map $g : Y \to Z$. Composing h with g gives a homotopy between $g \circ f_0$ and $g \circ f_1$. Precomposing also works: If

$g : W \to X$ is a map and $f_0, f_1 : X \to Y$ are homotopic, then $f_0 \circ g \simeq f_1 \circ g$. This lets us compose homotopy classes: we can complete the diagram:

$$\begin{array}{ccc} \mathbf{Top}(Y,Z) \times \mathbf{Top}(X,Y) & \longrightarrow & \mathbf{Top}(X,Z) \\ \downarrow & & \downarrow \\ [Y,Z] \times [X,Y] & \dashrightarrow & [X,Z] \end{array}$$

Definition 5.3. The *homotopy category* (of topological spaces) $\mathrm{Ho}(\mathbf{Top})$ has the same objects as $\mathbf{Top}$, but $\mathrm{Ho}(\mathbf{Top})(X,Y) = [X,Y] = \mathbf{Top}(X,Y)/\simeq$.

We may restate Theorem 5.2 as follows:

> For each n, the homology functor $H_n : \mathbf{Top} \to \mathbf{Ab}$ factors as $\mathbf{Top} \to \mathrm{Ho}(\mathbf{Top}) \to \mathbf{Ab}$; it is a "homotopy functor."

We will prove this in the next lecture, but let's stop now and think about some consequences.

Definition 5.4. A map $f : X \to Y$ is a *homotopy equivalence* if $[f] \in [X,Y]$ is an isomorphism in $\mathrm{Ho}(\mathbf{Top})$. In other words, there is a map $g : Y \to X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$.

Such a map g is a *homotopy inverse* for f; it is well-defined only up to homotopy.

Most topological properties are not preserved by homotopy equivalences. For example, compactness is not a homotopy-invariant property: Consider the inclusion $i : S^{n-1} \subseteq \mathbf{R}^n - \{0\}$. A homotopy inverse $p : \mathbf{R}^n - \{0\} \to S^{n-1}$ can be obtained by dividing a (always nonzero!) vector by its length. Clearly $p \circ i = 1_{S^{n-1}}$. We have to find a homotopy $i \circ p \simeq 1_{\mathbf{R}^n - \{0\}}$. This is a map $(\mathbf{R}^n - \{0\}) \times I \to \mathbf{R}^n - \{0\}$, and we can use $(v,t) \mapsto tv + (1-t)\frac{v}{||v||}$.

On the other hand:

Corollary 5.5. *Homotopy equivalences induce isomorphisms in homology.*

Proof. If f has homotopy inverse g, then f_* has inverse g_*. □

Definition 5.6. A space X is *contractible* if the map $X \to *$ is a homotopy equivalence.

Corollary 5.7. *Let X be a contractible space. The augmentation $\epsilon : H_*(X) \to \mathbf{Z}$ is an isomorphism.*

Homotopy equivalences in general may be somewhat hard to visualize. A particularly simple and important class of homotopy equivalences is given by the following definition.

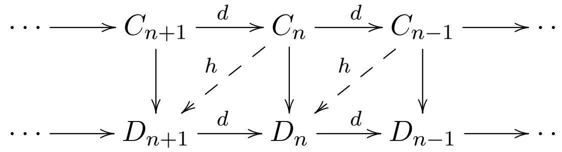
Definition 5.8. An inclusion $A \hookrightarrow X$ is a *deformation retract* provided that there is a map $h : X \times I \to X$ such that $h(x,0) = x$ and $h(x,1) \in A$ for all $x \in X$ and $h(a,t) = a$ for all $a \in A$ and $t \in I$.

For example, S^{n-1} is a deformation retract of $\mathbf{R}^n - \{0\}$.

We now set about constructing a proof of homotopy invariance of homology. The first step is to understand the analogue of homotopy on the level of chain complexes.

Definition 5.9. Let C_*, D_* be chain complexes, and $f_0, f_1 : C_* \to D_*$ be chain maps. A *chain homotopy* $h : f_0 \simeq f_1$ is a collection of homomorphisms $h : C_n \to D_{n+1}$ such that $dh + hd = f_1 - f_0$.

This relation takes some getting used to. It is an equivalence relation. Here's a picture (not a commutive diagram).



Lemma 5.10. *If $f_0, f_1 : C_* \to D_*$ are chain homotopic, then $f_{0*} = f_{1*} : H_*(C) \to H_*(D)$.*

Proof. We want to show that for every $c \in Z_n(C_*)$, the difference $f_1 c - f_0 c$ is a boundary. Well,

$$f_1 c - f_0 c = (dh + hd)c = dhc + hdc = dhc\,.$$

□

So homotopy invariance of homology will follow from

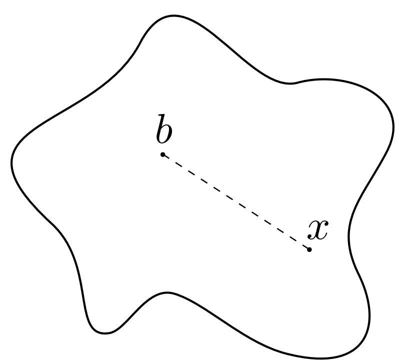
Proposition 5.11. *Let $f_0, f_1 : X \to Y$ be homotopic. Then $f_{0*}, f_{1*} : S_*(X) \to S_*(Y)$ are chain homotopic.*

To prove this we will begin with a special case.

Definition 5.12. A subset $X \subseteq \mathbf{R}^n$ is *star-shaped* with respect to $b \in X$ if for every $x \in X$ the interval

$$\{tb + (1-t)x : t \in [0,1]\}$$

lies in X.



Any nonempty convex region is star shaped. Any star-shaped region X is contractible: A homotopy inverse to $X \to *$ is given by sending $* \mapsto b$. One composite is perforce the identity. A homotopy from the other composite to the identity 1_X is given by $(x,t) \mapsto tb + (1-t)x$.

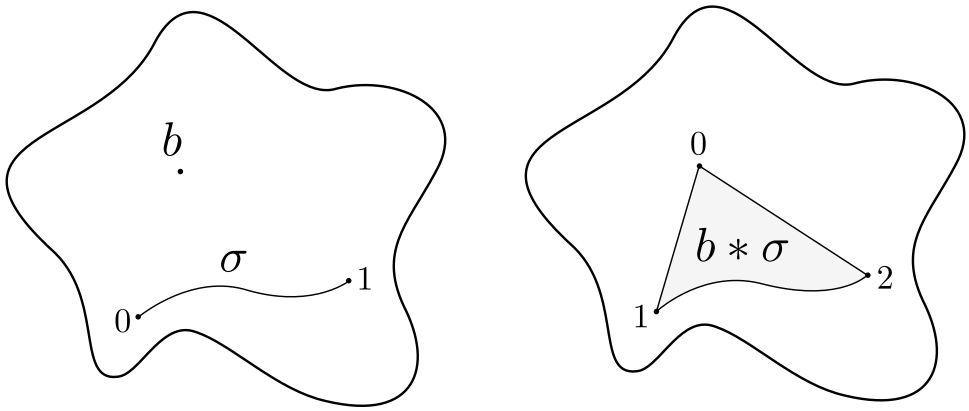
So we should expect that $\epsilon : H_*(X) \to \mathbf{Z}$ is an isomorphism if X is star-shaped. In fact, using a piece of language that the reader can interpret:

Proposition 5.13. *$S_*(X) \to \mathbf{Z}$ is a chain homotopy equivalence.*

Proof. We have maps $S_*(X) \xrightarrow{\epsilon} \mathbf{Z} \xrightarrow{\eta} S_*(X)$ where $\eta(1) = c_b^0$. Clearly $\epsilon\eta = 1$, and the claim is that $\eta\epsilon \simeq 1 : S_*(X) \to S_*(X)$. The chain map $\eta\epsilon$ concentrates everything at the point b: $\eta\epsilon\sigma = c_b^n$ for all $\sigma \in \mathrm{Sin}_n(X)$. Our chain homotopy $h : S_q(X) \to S_{q+1}(X)$ will actually send simplices to simplices. For $\sigma \in \mathrm{Sin}_q(X)$, define the chain homotopy evaluated on σ by means of the following "cone construction": $h(\sigma) = b * \sigma$, where

$$(b*\sigma)(t_0,\ldots,t_{q+1}) = t_0 b + (1-t_0)\sigma\left(\frac{(t_1,\ldots,t_{q+1})}{1-t_0}\right).$$

Explanation: The denominator $1-t_0$ makes the entries sum to 1, as they must if we are to apply σ to this vector. When $t_0 = 1$, this isn't defined, but it doesn't matter since we are multiplying by $1-t_0$. So $(b*\sigma)(1,0,\ldots,0) = b$; this is the vertex of the cone.



Setting $t_0 = 0$, we find

$$d_0 b * \sigma = \sigma\,.$$

Setting $t_i = 0$ for $i > 0$, we find

$$d_i b * \sigma = h d_{i-1}\sigma\,.$$

Using the formula for the boundary operator, we find

$$db*\sigma = \sigma - b * d\sigma$$

... *unless* $q = 0$, when

$$db*\sigma = \sigma - c_b^0\,.$$

This can be assembled into the equation

$$db * + b * d = 1 - \eta\epsilon$$

which is what we wanted. □

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